MAPS PRESERVING THE SPECTRUM OF GENERALIZED JORDAN PRODUCT OF OPERATORS

JINCHUAN HOU, CHI-KWONG LI, AND NGAI-CHING WONG

Abstract. Let $\mathcal{A}_1, \mathcal{A}_2$ be standard operator algebras on complex Banach spaces X_1, X_2, respectively. For $k \geq 2$, let $(i_1, \ldots, i_m)$ be a sequence with terms chosen from $\{1, \ldots, k\}$, and define the generalized Jordan product

$$T_1 \circ \cdots \circ T_k \;=\; T_{i_1} \cdots T_{i_m} + T_{i_m} \cdots T_{i_1}$$

on elements in $\mathcal{A}_i$. This includes the usual Jordan product $A_1 \circ A_2 = A_1A_2 + A_2A_1$, and the triple $\{A_1, A_2, A_3\} = A_1A_2A_3 + A_3A_2A_1$. Assume that at least one of the terms in $(i_1, \ldots, i_m)$ appears exactly once. Let a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ satisfy that

$$\sigma(\Phi(A_1) \circ \cdots \circ \Phi(A_k)) = \sigma(A_1 \circ \cdots \circ A_k)$$

whenever any one of $A_1, \ldots, A_k$ has rank at most one. It is shown in this paper that if the range of Φ contains all operators of rank at most three, then Φ must be a Jordan isomorphism multiplied by an mth root of unity. Similar results for maps between self-adjoint operators acting on Hilbert spaces are also obtained.

1. Introduction

There has been considerable interest in studying spectrum preserving maps on operator algebras in connection to the Kaplansky's problem on characterization of linear maps between Banach algebras preserving invertibility; see [16, 14, 3, 20, 2]. Early study focus on linear maps, additive maps, or multiplicative maps; see, e.g., [17]. Moreover, researchers considered maps preserving different types of spectra of operators such as the approximate spectrum, left invertible spectrum, right invertible spectrum, etc. Despite these variations, the maps often have the standard form

$$A \mapsto S^{-1}AS \qquad \text{or} \qquad A \mapsto S^{-1}A^*S$$

for a suitable invertible operator S, and A^* is the dual of A if A is a (bounded linear) operator between reflexive spaces. Many interesting techniques have been developed to derive these standard forms under different settings.

Date: October 6, 2009; a revision submitted to LAA.

2002 Mathematical Subject Classification. Primary 47B49; 47A12.

Key words and phrases. Standard operator algebra, spectral functions, Jordan triple-products of operators, skew products of operators, nonlinear-preserving problems.

Research of the first, second and third author was partially supported by NNSFC and PNSFS of China, by USA NSF and HK RCG, and by Taiwan NSC grant (NSC96-2115-M-110-004-MY3), respectively. The research began while the first author was visiting the College of William and Mary in the Fall of 2005 supported by the Freeman Foundation.

Recently, researchers have improved the results on spectrum preserving maps by showing that the map has the standard form under much weaker assumptions; see, e.g., [22, 21, 8, 9, 4, 7, 13]. For example, in [12], we characterize maps Φ (not assumed to be linear, additive or continuous) between standard operator algebras $\mathcal{A}_1, \mathcal{A}_2$ (not necessarily unital or closed) on complex Banach spaces X_1, X_2, respectively, such that $\sigma(\Phi(A_1)*\cdots*\Phi(A_k)) = \sigma(A_1*\cdots*A_k)$ whenever any one of A_i's is of rank at most one. Here, $T_1*\cdots*T_k = T_{i_1}\cdots T_{i_m}$ for a sequence $(i_1,\dots,i_m)$ with terms in $\{1,\dots,k\}$ such that one of the terms appears exactly once. Such product covers the usual product $T_1*\cdots*T_k = T_1\cdots T_k$, and the Jordan triple product $T_1*T_2 = T_2T_1T_2$. It is interesting to note that we can get the conclusion by requiring the spectrum preserving properties for low rank operators. In particular, we do not need to consider different types of spectra for such operators, as all of them coincide in this case. The list includes the left spectrum, the right spectrum, the boundary of the spectrum, the full spectrum, the point spectrum, the compression spectrum, the approximate point spectrum and the surjectivity spectrum, etc. Thus, our results in [12] unify and generalize several recent results of various spectrum preservers, see, e.g., [8, 9].

In this paper, we continue this line of study. In particular, we consider the generalized Jordan products of operators defined below.

Definition 1.1. Fix a positive integer k and a finite sequence $(i_1, i_2, \dots, i_m)$ such that $\{i_1, i_2, \dots, i_m\} = \{1, 2, \dots, k\}$ and there is an i_p not equal to i_q for all other q. Define a product for operators $T_1, \dots, T_k$ by

$$T_1 \circ \cdots \circ T_k = T_{i_1}\cdots T_{i_m} + T_{i_m}\cdots T_{i_1}.$$

Evidently, this definition covers the usual Jordan product $T_1T_2 + T_2T_1$, and the triple one: $\{T_1, T_2, T_3\} = T_1T_2T_3 + T_3T_2T_1$.

In the following, for $i = 1, 2$, let X_i be a complex Banach space, and $\mathcal{A}_i$ be a standard operator algebra on X_i, i.e., $\mathcal{A}_i$ contains all continuous finite rank operators on X_i. In particular, the Banach algebra $\mathcal{B}(X_i)$ of all bounded linear operators on X_i is a standard operator algebra. Note that we do not assume a standard operator algebra is unital or closed in any topology. Recall that a Jordan isomorphism $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ is either an inner automorphism or anti-automorphism. In this case, $\sigma(\Phi(A_1)\circ\cdots\circ\Phi(A_k)) = \sigma(A_1\circ\cdots\circ A_k)$ holds for all $A_1, \dots, A_k$. We will show that the converse is also true. It is interesting that consideration of low rank operators is again enough to ensure the conclusion of the converse statement.

Theorem 1.2. *Consider the product $T_1\circ\cdots\circ T_k$ defined in Definition 1.1. Suppose $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ satisfies*

$$\sigma(\Phi(A_1)\circ\cdots\circ\Phi(A_k)) = \sigma(A_1\circ\cdots\circ A_k). \tag{1.1}$$

whenever any of $A_1, \cdots, A_k$ has rank at most 1. Suppose also that the range of Φ contains all operators in $\mathcal{A}_2$ of rank at most 3. Then one of the following conditions holds.

(1) There exist a scalar λ with $\lambda^m = 1$ and an invertible operator T in $\mathcal{B}(X_1, X_2)$ such that

$$\Phi(A) = \lambda TAT^{-1} \quad \textit{for all } A \textit{ in } \mathcal{A}_1.$$

(2) The spaces X_1 and X_2 are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an invertible operator $T \in \mathcal{B}(X_1^*, X_2)$ such that

$$\Phi(A) = \lambda TA^*T^{-1} \quad \textit{for all } A \textit{ in } \mathcal{A}_1.$$

We remark that if the condition (1) or (2) in Theorem 1.2 holds, then Φ satisfies (1.1) for all $A_1, \ldots, A_k$ in $\mathcal{A}_1$. In fact, Φ preserves different kinds of spectra of $A_1 \circ \cdots \circ A_k$. For the generalized Jordan products of rank at most two appearing in (1.1), all such kinds of spectra coincide, however. So our results do unify, strengthen, and generalize several theorems in literature. See, e.g., [12, Remark 3.3]. Remark also that the linearity and continuity of Φ are parts of the conclusion. The proof of Theorem 1.2 is given in Section 3.

We also have a version for maps between the Jordan algebras of self-adjoint operators on Hilbert spaces, given in Section 4.

We note that our results are new even for the classical Jordan product $AB + BA$ and triple $ABC + CBA$. Similar to other papers, a crucial step in our proof is to show that the map Φ actually preserves rank one operators. To this end, we provide some new characterizations of rank one operators in term of the spectra of their Jordan products with rank one operators in Section 2. Nonetheless, the technique we employ in this paper is quite a bit different from those we usually see in the literature, e.g., [14, 11, 22, 21, 6, 8, 9, 4, 7, 13].

Finally, we would like to thank the Referee for his/her careful reading and helpful comments.

2. Characterizations of rank one operators

Lemma 2.1. *Suppose r and s are integers such that $s > r > 0$. Let A be a nonzero operator on a complex Banach space X of dimension at least three. The following conditions are equivalent.*

(a) A has rank one.

(b) $\sigma(B^rAB^s + B^sAB^r)$ has at most two distinct nonzero eigenvalues for any B in $\mathcal{B}(X)$.

(c) There does not exist an operator B with rank at most three such that $B^rAB^s + B^sAB^r$ has rank three and three distinct nonzero eigenvalues.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear.

To prove (c) $\Rightarrow$ (a), we consider the contrapositive. Suppose (a) is not true, i.e., A has rank at least 2.

If A has rank at least 3, then there are $x_1, x_2, x_3 \in X$ such that $\{Ax_1, Ax_2, Ax_3\}$ is linearly independent. Consider the operator matrix of A on the span of $\{x_1, x_2, x_3, Ax_1, Ax_2, Ax_3\}$

and its complement:

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}.$$

Then $A_{11} \in M_n$ with $3 \le n \le 6$. By [12, Lemma 2.3], there is a nonsingular U on the span of $\{x_1, x_2, x_3, Ax_1, Ax_2, Ax_3\}$ such that $U^{-1}A_{11}U$ has an invertible 3-by-3 leading submatrix. We may further assume that the 3-by-3 matrix is in triangular form with nonzero diagonal entries a_1, a_2, a_3. Now let B in $\mathcal{A}$ have operator matrix

$$\begin{pmatrix} B_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where $UB_{11}U^{-1} = \operatorname{diag}(1, b_2, b_3) \oplus 0_{n-3}$ with B_{11} using the same basis as that of A_{11} and b_2, b_3 being chosen such that $a_1, a_2b_2^{r+s}, a_3b_3^{r+s}$ are three distinct nonzero numbers. It follows that $B^rAB^s + B^sAB^r$ has rank 3 with three distinct nonzero eigenvalues.

Next, suppose A has rank 2. Choosing a suitable space decomposition of X, we may assume that A has operator matrix $A_1 \oplus 0$, where A_1 has one of the following form.

$$\text{(i)} \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(iv)} \begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix}.$$

If (i) holds, set $\theta = \pi/s$. Then $\cos r\theta \neq \pm 1$ and $\cos r\theta \neq \pm\sqrt{\cos 2r\theta}$. Let $d > 0$ such that $a(\cos r\theta \pm \sqrt{\cos 2r\theta}), -2cd^{r+s}$ are three distinct nonzero numbers. Let $B \in \mathcal{A}$ be represented by the operator matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & d \end{pmatrix} \oplus 0.$$

Then $B^s = -I_2 \oplus [d^s] \oplus 0$, and $-(B^rAB^s + B^sAB^r)$ has operator matrix

$$\begin{pmatrix} 2a\cos r\theta & -a\sin r\theta & * \\ a\sin r\theta & 0 & * \\ 0 & 0 & -2cd^{r+s} \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues $a(\cos r\theta \pm \sqrt{\cos 2r\theta}), -2cd^{r+s}$.

Suppose (ii) holds. Let $d > 0$ be such that $2ad^{r+s}$, $s + r \pm 2\sqrt{rs}$ are three distinct nonzero numbers. Then construct B by the operator matrix

$$\begin{pmatrix} d & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \oplus 0.$$

Then $B^rAB^s + B^sAB^r$ has operator matrix

$$\begin{pmatrix} 2ad^{r+s} & 0 & 0 \\ 0 & s+r & 2 \\ 0 & 2rs & s+r \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues $2ad^{r+s}, s+r \pm 2\sqrt{rs}$.

Suppose (iii) holds. First, assume that $s = 2r$. Let B be such that B^r has operator matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \oplus 0.$$

Then $B^rAB^s + B^sAB^r$ has operator matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues: $2^{1/3}, 2^{1/3}e^{i2\pi/3}, 2^{1/3}e^{i4\pi/3}$.

Next, suppose $s/r \neq 2$. Then $s > 2$ and $2r/s$ is not an integer. Let $\theta_1 = 2\pi/s$, $\theta_2 = 4\pi/s$. Then $1, e^{ir\theta_1}, e^{ir\theta_2}$ are distinct because $e^{i4\pi r/s} = e^{i2\pi(2r/s)} \neq 1$ and $e^{ir\theta_1} = e^{ir\theta_2}/e^{ir\theta_1} = e^{i2\pi r/s} \neq 1$. Thus, there exists an invertible $S \in M_3$ such that

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{ir\theta_1} & 0 \\ 0 & 0 & e^{ir\theta_2} \end{pmatrix} = S^{-1} \begin{pmatrix} 1 & 0 & 0 \\ 1 & e^{ir\theta_1} & 0 \\ 0 & 2 & e^{ir\theta_2} \end{pmatrix} S.$$

Let B have operator matrix

$$S \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{i\theta_1} & 0 \\ 0 & 0 & e^{i\theta_2} \end{pmatrix} S^{-1} \oplus 0.$$

The operator matrix $B^s = I_3 \oplus 0$ and the operator matrix of B^r has the form

$$S \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{ir\theta_1} & 0 \\ 0 & 0 & e^{ir\theta_2} \end{pmatrix} S^{-1} \oplus 0 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & e^{ir\theta_1} & 0 \\ 0 & 2 & e^{ir\theta_2} \end{pmatrix} \oplus 0.$$

Then $B^rAB^s + B^sAB^r = AB^r + B^rA$ has operator matrix

$$\begin{pmatrix} 1 & 1+e^{ir\theta_1} & 0 \\ 0 & 3 & e^{ir\theta_1} + e^{ir\theta_2} \\ 0 & 0 & 2 \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues.

If (iv) holds, then X has dimension at least 4. We may use a different decomposition of X and assume that A has operator matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \oplus 0.$$

Let $\theta = \pi/(2(r+s))$ and $d > 0$ be such that $1 \pm \sqrt{\sin(2r\theta)\sin(2s\theta)}$ and d^{r+s} are 3 distinct nonzero numbers, and let B be an operator in $\mathcal{B}(X)$ such that B^ℓ has operator matrix

$$B^\ell = \begin{pmatrix} \cos\ell\theta & -\sin\ell\theta & 0 \\ \sin\ell\theta & \cos\ell\theta & 0 \\ 0 & 0 & d^\ell \end{pmatrix} \oplus 0$$

for any positive integer ℓ. Then $B^rAB^s + B^sAB^r$ has operator matrix

$$\begin{pmatrix} \sin((r+s)\theta) & 2\cos r\theta\cos s\theta & 0 \\ 2\sin r\theta \sin s\theta & \sin((r+s)\theta) & 0 \\ 0 & 0 & d^{r+s} \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues. ■

Lemma 2.2. *Suppose s is a positive integer. Let X be a complex Banach space of dimension at least three. Let $A \in \mathcal{B}(X)$ be such that $A^2 \neq 0$. Then the following are equivalent.*

(a) A has rank one.

(b) $\sigma(AB^s + B^sA)$ has at most two distinct nonzero eigenvalues whenever $\operatorname{rank}(B) \leq 3$ *and* $\operatorname{rank}(AB^s + B^sA) \leq 3$.

Proof. One direction is trivial. Suppose A has rank at least 2 such that $A^2 \neq 0$. First assume that A has rank 2. Choosing a suitable decomposition of X, we may assume that A has operator matrix $A_1 \oplus 0$, where A_1 has one of the following form

$$\text{(i)} \begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(iii)} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \text{(iv)} \begin{pmatrix} 0_2 & I_2 \\ 0_2 & 0_2 \end{pmatrix}.$$

Since $A^2 \neq 0$, (iv) is impossible. If (i) holds, set $\theta = \pi/(2s+1)$ so that $\cos s\theta \neq \pm\sqrt{\cos 2s\theta}$. Let $d > 0$ such that $a(\cos s\theta \pm \sqrt{\cos 2s\theta}), 2cd^s$ are three distinct nonzero numbers. Let B have operator matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & d \end{pmatrix} \oplus 0.$$

Then similar to the proof of Lemma 2.1, we see that $AB^s + B^sA$ has operator matrix

$$\begin{pmatrix} 2a\cos s\theta & -a\sin s\theta & * \\ a\sin s\theta & 0 & * \\ 0 & 0 & 2cd^s \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues $a(\cos s\theta \pm \sqrt{\cos 2s\theta}), 2cd^{r+s}$.

Suppose (ii) holds. Let $d > 0$ be such that $2d, d \pm \sqrt{a^2+d^2}$ are three distinct nonzero numbers. Since the matrix

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 2d & 2 \end{pmatrix}$$

is similar to a matrix with distinct eigenvalues $-1, 1, 2$, there exists an operator B of rank 3 such that the operator matrix of B^s equals $C \oplus 0$. It follows that the operator matrix of $AB^s + B^sA$ is

$$\begin{pmatrix} 0 & a & 1 \\ a & 2d & 2 \\ 0 & 0 & 2d \end{pmatrix} \oplus 0,$$

which has rank 3 and distinct nonzero eigenvalues $2d, d \pm \sqrt{a^2+d^2}$.

Suppose (iii) holds. Since the matrix

$$C = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 2 & 2 \end{pmatrix}$$

has distinct eigenvalues $0, 1, 2$, there exists an operator B of rank 2 such that the operator matrix of B^s equals $C \oplus 0$. Then $AB^s + B^sA$ has operator matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{pmatrix} \oplus 0,$$

which has rank 3 with three distinct nonzero eigenvalues $1, 2, 3$.

Now, suppose A has rank at least 3. Since $A^2 \neq 0$, there is $x \in X$ such that $A^2x \neq 0$. We consider 3 cases.

Case 1. There is $x \in X$ such that $[x, Ax, A^2x]$ has dimension 3. Decompose X into $[x, Ax, A^2x]$ and its complement. The operator matrix of A has the form

$$\begin{pmatrix} 0 & 0 & c_1 & * \\ 1 & 0 & c_2 & * \\ 0 & 1 & c_3 & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Note that for $t > 0$, the matrix

$$C = \begin{pmatrix} 2t & 1 & 0 \\ 0 & t & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

has three distinct eigenvalues: $2t, t, 0$. So, there is B_1 of rank 2 such that $B_1^s = C$. Let B have operator matrix $B_1 \oplus 0$. Then $AB^s + B^sA$ has operator matrix $\begin{pmatrix} tR_1 + R_2 & * \\ 0 & 0 \end{pmatrix}$, where

$$R_1 = \begin{pmatrix} 0 & 0 & 2c_1 \\ 3 & 0 & c_2 \\ 0 & 1 & 0 \end{pmatrix} \qquad \text{and} \qquad R_2 = \begin{pmatrix} 1 & 0 & c_2 \\ 0 & 3 & 2c_2 \\ 0 & 0 & 2 \end{pmatrix}.$$

Since R_2 has distinct eigenvalues $1, 2, 3$, the matrix $tR_1 + R_2$ will have three distinct nonzero eigenvalues for sufficiently small t. Hence, $AB^s + B^sA$ has rank 3 with three distinct nonzero eigenvalues.

Case 2. Suppose Case 1 does not hold, and there is $x \in X$ such that $A^2x \neq 0$ and $[x, Ax, A^2x]$ has dimension 2. Clearly, we cannot have $Ax = \lambda x$. Otherwise, $[x, Ax, A^2x]$ has dimension 1. Hence, $A^2x = b_1x + b_2Ax$ so that $(b_1, b_2) \neq (0, 0)$. Since A has rank at least three, there is $y \in X$ such that $Ay \notin [x, Ax]$. We claim that there is a decomposition of X so that A has operator matrix

$$\begin{pmatrix} A_0 & * \\ 0 & * \end{pmatrix}, \tag{2.1}$$

where $A_0 \in M_3$ is in upper triangular form of rank at least 2 and with at least one nonzero eigenvalue.

To prove our claim, suppose $Ay = c_1x + c_2Ax + c_3y$ with $c_3 \neq 0$. Using $[x, Ax, y]$ and its complement, the operator matrix of A has the form

$$\begin{pmatrix} A_1 & * \\ 0 & * \end{pmatrix} \qquad \text{with} \qquad A_1 = \begin{pmatrix} 0 & b_1 & c_1 \\ 1 & b_2 & c_2 \\ 0 & 0 & c_3 \end{pmatrix},$$

where A_1 has rank at least 2. Since $(b_1, b_2) \neq (0, 0)$, the matrix A_1 has at least two nonzero eigenvalues including c_3. We may replace $\{x, Ax, y\}$ by a linearly independent family $\{\hat{x}_1, \hat{x}_2, \hat{x}_3\}$ in $[x, Ax, y]$ so that the operator matrix of A has the form described in (2.1).

Next, suppose $Ay \notin [x, Ax, y]$. Note that $[y, Ay, A^2y]$ has dimension 2 by our assumption in Case 2. In this subcase, $Ay \neq \lambda y$. So, $A^2y = d_1y + d_2Ay$ with $(d_1, d_2) \neq (0, 0)$. With respect to $[x, Ax, y, Ay]$ and its complement in X, the operator matrix of A has the form

$$\begin{pmatrix} A_2 & * \\ 0 & * \end{pmatrix} \qquad \text{with} \qquad A_2 = \begin{pmatrix} 0 & b_1 \\ 1 & b_2 \end{pmatrix} \oplus \begin{pmatrix} 0 & d_1 \\ 1 & d_2 \end{pmatrix}.$$

Since $(b_1, b_2) \neq (0,0)$ and $(d_1, d_2) \neq (0,0)$, A_2 has rank at least 2 and at least 2 nonzero eigenvalues. We may choose an independent family $\{\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4\}$ in $[x, Ax, y, Ay]$ so that the operator matrix of A_2 with respect to $[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4]$ is in upper triangular form, whose leading 3-by-3 submatrix A_0 has rank at least 2 and has at least one nonzero eigenvalue. So, the operator matrix of A with respect to $[\tilde{x}_1, \tilde{x}_2, \tilde{x}_3]$ and its complement has the form described in (2.2). So, our claim is verified.

Now, if A_0 in (2.2) is invertible, then there is B with operator matrix $B_1 \oplus 0$, where $B_1 = \operatorname{diag}(1, b_2, b_3)$, and $AB^s + B^sA$ has operator matrix

$$\begin{pmatrix} A_0B_1^s + B_1^sA_0 & * \\ 0 & 0 \end{pmatrix},$$

which has rank 3 with three distinct nonzero eigenvalues. Suppose A_0 is singular. Since A_0 in (2.2) has rank two and at least one nonzero eigenvalue, we may assume that A_0 has the forms

$$\begin{pmatrix} a & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & c \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

In each case, we can use the arguments in the proof when A has rank 2 to choose B with operator matrix $B_1 \oplus 0$ so that $B_1 \in M_3$ and $AB^s + B^sA$ has operator matrix

$$\begin{pmatrix} A_0B_1^s + B_1^sA_0 & * \\ 0 & 0 \end{pmatrix},$$

which is a rank 3 operator with three distinct nonzero eigenvalues.

Case 3. Suppose $[x, Ax, A^2x]$ has dimension one for any nonzero x in X. Then A is a scalar operator. Let B have operator matrix $\operatorname{diag}(1,2,3) \oplus 0$. Then $AB^s + B^sA$ has rank 3 and three distinct nonzero eigenvalues. ∎

Corollary 2.3. *Suppose s is a positive integer. Let X be a complex Banach space X of dimension at least three, and let A in $\mathcal{B}(X)$ be nonzero. The following conditions are equivalent.*

(a) A has rank one, or A has rank two such that $A^2 = 0$.

(b) $\sigma(AB^s + B^sA)$ has at most two distinct nonzero eigenvalues for any B in $\mathcal{B}(X)$.

(c) There does not exist an operator B with rank at most three such that $AB^s + B^sA$ has rank at most six with three distinct nonzero eigenvalues.

Proof. (a) $\Rightarrow$ (b). If A has rank one, then (b) clearly holds. If A has rank two and $A^2 = 0$, then there is a decomposition of X such that A has operator matrix

$$\begin{pmatrix} 0_2 & I_2 & 0 \\ 0_2 & 0_2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So, for any B in $\mathcal{A}$ such that B^s has operator matrix

$$\begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix},$$

$AB^s + B^sA$ has operator matrix

$$\begin{pmatrix} B_{21} & B_{22}+B_{11} & B_{23} \\ 0 & B_{21} & 0 \\ 0 & B_{31} & 0 \end{pmatrix},$$

whose nonzero eigenvalues are the same as those of $B_{21} \in M_2$. Thus, there are at most two nonzero distinct eigenvalues.

The implication (b) $\Rightarrow$ (c) is clear.

Finally, we verify the implication (c) $\Rightarrow$ (a). If (c) holds, by Lemma 2.2, we see that A is either rank 1 or $A^2 = 0$. If $A^2 = 0$, we claim that A has rank at most 2. If it is not true, then we can find x_1, x_2, x_3 in X such that $\{Ax_1, Ax_2, Ax_3\}$ is linearly independent. Then with respect to $[x_1, x_2, x_3, Ax_1, Ax_2, Ax_3]$ and its complement, the operator matrix of A has the form

$$\begin{pmatrix} 0_3 & 0_3 & * \\ I_3 & 0_3 & * \\ 0 & 0 & * \end{pmatrix}.$$

Let $B \in \mathcal{B}(X)$ have rank 3 with three distinct nonzero eigenvalues such that B^s has operator matrix

$$\begin{pmatrix} D & D \\ 0_3 & 0_3 \end{pmatrix} \oplus 0, \qquad \text{with} \qquad D = \operatorname{diag}(1,2,3).$$

Then $AB^s + B^sA$ has rank 6 and 3 distinct eigenvalues. Our conclusion follows. ∎

3. Maps preserving spectrum of generalized Jordan products of low rank

Theorem 1.2 clearly follows from the special case below, by considering $A_{i_p} = A$ and all other $A_{i_q} = B$.

Theorem 3.1. *Suppose a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ between standard operator algebras satisfies*

$$\sigma(\Phi(B)^r\Phi(A)\Phi(B)^s + \Phi(B)^s\Phi(A)\Phi(B)^r) = \sigma(B^rAB^s + B^sAB^r), \tag{3.1}$$

whenever A or B has rank at most one. Suppose also that the range of Φ contains all operators in $\mathcal{A}_2$ of rank at most 3. Then one of the two assertions in Theorem 1.2 holds with $m = r + s + 1$.

We note that the case when $s = r > 0$ has been verified in [12]. So, unless specified otherwise, we will assume $s > r \geq 0$ in the rest of this section. In below, we first show that Φ in Theorem 3.1 is injective.

For a Banach space X denote by $\mathcal{I}_1(X)$ the set of all rank one idempotent operators in $\mathcal{B}(X)$. In other words, $\mathcal{I}_1(X)$ consists of all bounded operators $x \otimes f$ with $x \in X$, $f \in X^*$ and $\langle x, f\rangle = f(x) = 1$.

Lemma 3.2. *Let $A, A' \in \mathcal{B}(X)$ for some Banach space X. Suppose*

$$\langle Ax, f\rangle = 0 \quad \textit{if and only if} \quad \left\langle A'x, f\right\rangle = 0, \quad \forall x \otimes f \in \mathcal{I}_1(X).$$

Then $A' = \lambda A$ for some scalar λ.

Proof. First suppose there is a nonzero x in X such that $Ax = \alpha x$ for some nonzero scalar α. Then for any f in X^* with $\langle x, f\rangle \neq 0$, we have $\langle Ax, f\rangle \neq 0$, and thus $\langle A'x, f\rangle \neq 0$. Hence, $A'x = \beta x$ for some nonzero scalar β, and $Ax, A'x$ are linearly dependent.

Then suppose $\{x, Ax\}$ is linearly independent. Choose any $x \otimes f$ in $\mathcal{I}_1(X)$ with $\langle Ax, f\rangle = 0$. Then for any g in X^* with $\langle x, g\rangle = 0$, we have $\langle x, f + g\rangle = 1$. If $\langle Ax, g\rangle = 0$ then $\langle Ax, f + g\rangle = 0$, and thus $\langle A'x, f + g\rangle = 0$. This eventually gives $\langle A'x, g\rangle = 0$. Thus, together with the assumption, we see that $Ax, A'x$ are linearly dependent again.

If A has rank one then the assertion is plain. Assume Ax, Ay are linearly independent for some x, y in X. Then $A'x = \lambda_x Ax$, $A'y = \lambda_y Ay$ and $A'(x + y) = \lambda_{x+y} A(x + y)$ for some scalars λ_x, λ_y and λ_{x+y}. This forces $\lambda_x = \lambda_y = \lambda_{x+y}$. So the assertion follows. ■

Lemma 3.3. *Suppose r and s are nonnegative integers with $(r, s) \neq (0, 0)$. Let X be a complex Banach space. If $A, A' \in \mathcal{B}(X)$ satisfy*

$$\sigma(B^r AB^s + B^s AB^r) = \sigma(B^r A'B^s + B^s A'B^r), \quad \forall B \in \mathcal{I}_1(X),$$

then $A = A'$.

Proof. We may suppose that $A' \neq 0$ since it is obvious that $\sigma(B^r AB^s + B^s AB^r) = \{0\}$ for all rank one idempotents B implies that $A = 0$.

Assume first that $s \geq r > 0$. Then the assumption implies that $\sigma(BAB) = \sigma(BA'B)$ and hence $f(Ax) = \mathrm{tr}(BAB) = \mathrm{tr}(BA'B) = f(A'x)$ for all rank one idempotents $B = x \otimes f$. By Lemma 3.2, we see that $A' = A$.

Assume then that $s > r = 0$ and write the rank-one idempotent B in the form $B = x \otimes f$ with $\langle x, f\rangle = 1$. Then $AB^s + B^s A = AB + BA$, and either

(i) $\mathrm{tr}\,(AB + BA)$ is the sum of the elements in $\sigma(AB + BA)$, or

(ii) $AB + BA$ has rank two and a repeated nonzero eigenvalue so that $\mathrm{tr}\,(AB + BA)$ is twice the sum of the elements in $\sigma(AB + BA)$.

Therefore, $\mathrm{tr}\,(AB + BA) = 0$ if and only if $\sigma(AB + BA) = \{0\}$ or $\{\alpha, -\alpha, 0\}$ for some nonzero α. Since $\sigma(AB + BA) = \sigma(A'B + BA')$, we see that $\mathrm{tr}\,(AB + BA) = 0$ if and only if

$\mathrm{tr}\,(A'B + BA') = 0$. It follows from Lemma 3.2 again that $A' = \lambda A$ for some scalar λ. But the spectrum coincidence implies $\lambda = 1$. ■

As a direct consequence of Lemma 3.3 and the condition (3.1), we have

Corollary 3.4. *Let Φ satisfy the hypothesis of Theorem 3.1. Then Φ is injective, and $\Phi(0) = 0$.*

In the following, we present the proof of Theorem 3.1 in several steps.

3.1. **The case $\dim X_2 = 1$.** We claim $\dim X_1 = 1$. Suppose on contrary that $\dim X_1 \geq 2$. Let $\Phi(A) = \lambda_A \in \mathbb{C}$. Then for the rank one idempotent $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$ in $\mathcal{A}_1$ we have by (3.1) that $\lambda_B^{r+s+1} = 1$. Moreover,

$$\sigma(B^s AB^r + B^r AB^s) = \sigma(2\lambda_A \lambda_B^{r+s}), \quad \forall A \in \mathcal{A}_1.$$

If $r = 0$ then $BA + AB = \begin{pmatrix} 2a & b \\ c & 0 \end{pmatrix} \oplus 0$ for any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \oplus 0$ in $\mathcal{A}_1$. In particular, $BA + AB$ can have two distinct eigenvalues for some choices of a, b, c. This contradiction forces $\dim X_1 = 1$. If $r > 0$ then we will have

$$\mathrm{tr}\,(BAB) = \lambda_A \lambda_B^{r+s}, \quad \forall A \in \mathcal{A}_1.$$

Thus

$$\Phi(A) = \lambda_A = \lambda_B \mathrm{tr}\,(BAB), \quad \forall A \in \mathcal{A}_1.$$

Using another rank one idempotent B' in place of B we will have the same conclusion. Hence,

$$\lambda_B \mathrm{tr}\,(BAB) = \lambda_{B'} \mathrm{tr}\,(B'AB'), \quad \forall A \in \mathcal{A}_1.$$

This is possible only when $\dim X_1 = 1$. In both cases, we see that $\Phi : \mathbb{C} \to \mathbb{C}$ is an algebra isomorphism given by $\Phi(\alpha) = \lambda\alpha$ with $\lambda^{r+s+1} = 1$.

3.2. **The case $\dim X_2 = 2$.** We first claim that $\dim X_1 \geq 2$. Suppose on contrary that $\dim X = 1$. Write $\Phi(\alpha) = A_\alpha$. By (3.1),

$$\sigma(A_\beta^s A_\alpha A_\beta^r + A_\beta^r A_\alpha A_\beta^s) = \{2\alpha\beta^{r+s}\}, \quad \forall \alpha, \beta \in \mathbb{C}.$$

By the surjectivity of Φ, we assume $A_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. Then A_α^{r+s+1} has two distinct eigenvalues 1 and 2^{r+s+1}, a contradiction.

The following lemma verifies Theorem 3.1 for the case when $\dim X_2 = 2$. Indeed, similar arguments can be used to study the cases when $2 \leq \dim X_2 \leq \dim X_1 < \infty$. Anyway, we will use a unified arguments for all the cases when $\dim X_2 \geq 3$ in the next subsection.

Lemma 3.5. *Let $n \geq 2$ be a cardinal number. Denote by $\mathcal{V}_n$ either a standard operator algebra on a Banach space of dimension n, or the Jordan algebra of all self-adjoint bounded operators on a Hilbert space of dimension n. Denote by M_2 the algebra of all 2×2 matrices. Let $\Phi : \mathcal{V}_n \to M_2$ satisfy*

$$\sigma(B^r AB^s + B^s AB^r) = \sigma(\Phi(B)^r \Phi(A)\Phi(B)^s + \Phi(B)^s \Phi(A)\Phi(B)^r) \tag{3.2}$$

whenever A and B in $\mathcal{V}_n$ have rank one. Then $n = 2$, and there is an mth root of unity, λ, and an invertible operator S such that Φ assumes either the form

$$\Phi(X) = \lambda S^{-1} XS \quad \text{or} \quad \Phi(X) = \lambda S^{-1} X^t S.$$

Proof. We first note that $\mathcal{V}_n$ contains a copy of $\mathcal{V}_2$. So we can assume that Φ is a map from $\mathcal{V}_2$ into M_2. Let A be a rank one orthogonal projection. Then $B \in \mathcal{V}_2$ satisfies

$$[\mathrm{tr}\,(A^r BA^s + A^s BA^r)]^2 \neq 4 \det(A^r BA^s + A^s BA^r)$$

if and only if $A^r BA^s + A^s BA^r$ has distinct eigenvalues. Thus, the set $\mathcal{S}$ of all such matrices B form an open dense set of $\mathcal{V}_2$. Thus, for four linearly independent rank one orthonormal projections A_1, A_2, A_3, A_4, we get a dense set $\mathcal{S}$ of matrices $B \in V_2$ such that $A_j^r BA_j^s + A_j^r BA_j^s$ has two distinct eigenvalues for $j = 1, \ldots, 4$. For each $B \in \mathcal{S}$, the rank at most two operator $A^r BA^s + A^s BA^r$ has two distinct eigenvalues, and so is $\Phi(A)^r \Phi(B)\Phi(A)^s + \Phi(A)^s \Phi(B)\Phi(A)^r$ for all $A \in \{A_1, \ldots, A_4\}$ and $B \in \mathcal{S}$. It follows that for $m = r + s + 1$

$$\begin{aligned} 2\mathrm{tr}\,(AB) &= 2\mathrm{tr}\,(A^{m-1}B) = \mathrm{tr}\,(A^r BA^s + A^s BA^r) \\ &= \mathrm{tr}\,(\Phi(A)^r \Phi(B)\Phi(A)^s + \Phi(A)^s \Phi(B)\Phi(A)^r) = 2\mathrm{tr}\,(\Phi(A)^{m-1}\Phi(B)) \end{aligned}$$

for all $A \in \{A_1, \ldots, A_4\}$ and $B \in \mathcal{S}$. For $X = (x_{ij}) \in M_2$, let $v(X) = (x_{11}\ x_{12}\ x_{21}\ x_{22})^t$. Form the 4×4 matrices

$$R = [v(A_1)|v(A_2)|v(A_3)|v(A_4)]^t$$

and

$$\hat{R} = [v(\Phi(A_1)^m)|v(\Phi(A_2)^m)|v(\Phi(A_3)^m)|v(\Phi(A_4)^m)]^t.$$

Then

$$Rv(B^t) = \hat{R}v(\Phi(B^t)), \qquad \text{for all } B \in \mathcal{S}.$$

Pick a linearly independent set $\{B_1, B_2, B_3, B_4\}$ in $\mathcal{S}$. If

$$T = [v(B_1)|v(B_2)|v(B_3)|v(B_4)]$$

and

$$\hat{T} = [v(\Phi(B_1))|v(\Phi(B_2))|v(\Phi(B_3))|v(\Phi(B_4))],$$

then

$$RT = \hat{R}\hat{T}.$$

Since the left side is the product of two invertible matrices, the two matrices on the right side are invertible. So, $\hat{R}^{-1}Rv(B) = v(\Phi(B))$ for all $B \in \mathcal{S}$. Consider the linear map $\hat{\Phi} : \mathcal{V}_2 \to M_2$ such that

$$\hat{R}^{-1}Rv(B) = v(\hat{\Phi}(B)).$$

Then

$$\sigma(A^rBA^s + A^sBA^r) = \sigma(\hat{\Phi}(A)^r\hat{\Phi}(B)\hat{\Phi}(A)^s + \hat{\Phi}(A)^s\hat{\Phi}(B)\hat{\Phi}(A)^r)$$

for all $A, B \in \mathcal{S}$. By the continuity of $X \mapsto \sigma(X)$, we see that the set equality holds for all $A, B \in \mathcal{V}_2$. Let $A = B$ be a rank one orthogonal projection. Since $\sigma(A^{m+1}) = \sigma(\hat{\Phi}(A)^{m+1})$, we see that $\hat{\Phi}(A)$ is similar to $\lambda\mathrm{diag}(1, 0)$ with $\lambda^{m+1} = 1$. By a connectedness argument, we see that such λ is the same for every rank one orthogonal projection. Dividing Φ by λ, we can assume $\lambda = 1$. By Lemma 3.3, we see that $\hat{\Phi}$ sends exactly zero to zero. In case A is a rank one square zero matrix, $\sigma(\hat{\Phi}(A)^m) = \sigma(A^m) = \{0\}$, and thus $\hat{\Phi}(A)$ is also a rank one square zero matrix.

Write every invertible self-adjoint matrix A in $\mathcal{V}_2$ as a linear sum of two orthogonal rank one projections. By (3.2), we see that $\hat{\Phi}$ sends orthogonal rank one projections to orthogonal rank one projections. Hence $\hat{\Phi}(A^2) = \hat{\Phi}(A)^2$ for all self-adjoint 2×2 matrices. It follows that $\hat{\Phi}(AB + BA) = \hat{\Phi}(A)\hat{\Phi}(B) + \hat{\Phi}(B)\hat{\Phi}(A)$ for all self-adjoint 2×2 matrices. If $\mathcal{V}_2 = M_2$ then $\hat{\Phi}((A+iB)^2) = \hat{\Phi}(A^2) + i\hat{\Phi}(AB+BA) + \hat{\Phi}(B^2) = \hat{\Phi}(A+iB)^2$, whenever A, B are self-adjoint 2×2 matrices. Consequently, $\hat{\Phi}$ has the standard form $X \mapsto S^{-1}XS$ or $X \mapsto S^{-1}X^tS$, where S is an invertible 2×2 matrix. Note that $\Phi(X) = \hat{\Phi}(X)$ for all $X \in \mathcal{S}$. We may modify f and assume that $\Phi(X) = X$ for all $X \in \mathcal{S}$. So, for any $X \in \mathcal{V} \setminus \mathcal{S}$,

$$\sigma(B^rXB^s + B^sXB^r) = \sigma(B^r\Phi(X)B^s + B^s\Phi(X)B^r)$$

for all $B \in \mathcal{S}$. One can then argue that $\Phi(X) = X$ by Lemma 3.3. Finally, by Corollary 3.4 we see that Φ is injective, and thus $n = 2$. ∎

3.3. **The case** $\dim X_2 \geq 3$**.** Here are some technical lemmas.

Lemma 3.6. *Let X be a complex Banach space and $A \in \mathcal{B}(X)$. Assume that $x \otimes f \in \mathcal{B}(X)$ is a rank one idempotent. Then the at most rank two operator $A(x \otimes f) + (x \otimes f)A$ has*

(1) a nonzero repeated eigenvalue if and only if $\langle Ax, f\rangle \neq 0$ and $\langle A^2x, f\rangle = 0$;

(2) two distinct nonzero eigenvalues if and only if $\langle A^2x, f\rangle \neq 0$ and $\langle A^2x, f\rangle \neq \langle Ax, f\rangle^2$.

Proof. (1) Assume that $B = A(x\otimes f)+(x\otimes f)A = Ax\otimes f + x\otimes A^*f$ has rank two and a nonzero repeated eigenvalue λ. Then $\langle Ax, f\rangle = \frac{1}{2}\mathrm{tr}(A(x \otimes f) + (x \otimes f)A) = \lambda \neq 0$. Furthermore, let $u = Ax - \lambda x$ and $g = A^*f - \lambda f$. Then $\langle x, g\rangle = \langle u, f\rangle = 0$. In a space decomposition with basic vectors u, x, the operator B has a matrix form

$$B = \begin{pmatrix} 0 & 1 \\ \langle u, g\rangle & 2\lambda \end{pmatrix} \oplus 0.$$

Hence, the spectrum of B contains the zeros of $t^2 - 2\lambda t - \langle u, g\rangle$, which gives the repeated eigenvalue λ of the operator. We have $\langle u, g\rangle = -\lambda^2$. So, $\langle A^2x, f\rangle = \langle Ax, A^*f\rangle = \lambda^2 + \langle u, g\rangle = 0$.

Conversely, if $\langle Ax, f\rangle = \lambda \neq 0$ and $\langle A^2x, f\rangle = 0$, then $Ax = \lambda x + u$ and $A^*f = \lambda f + g$ with $\langle u, f\rangle = \langle x, g\rangle = 0$ and $\langle u, g\rangle = -\lambda^2$. This implies that λ is a repeated nonzero eigenvalue of $Ax \otimes f + x \otimes A^*f$.

(2) Use the same notations as in the proof of (1). If $A(x \otimes f) + (x \otimes f)A$ has two distinct nonzero eigenvalues, then, by (1), $\langle A^2x, f\rangle = \langle Ax, A^*f\rangle = \lambda^2 + \langle u, g\rangle = \langle Ax, f\rangle^2 + \langle u, g\rangle \neq 0$ and $\langle u, g\rangle \neq 0$. Thus, $\langle A^2x, f\rangle \neq \langle Ax, f\rangle^2$. The converse is clear. ∎

Lemma 3.7. *Let X be a complex Banach space of dimension at least two, and let $A_i \in \mathcal{B}(X)$ with $A_i^2 \neq 0$, $i = 1, 2, 3$. Then, the set of rank one idempotent operators $P \in \mathcal{B}(X)$ satisfying that every $A_iP + PA_i$, $i = 1, 2, 3$, has two distinct nonzero eigenvalues is dense in the set of all rank one idempotents in $\mathcal{B}(X)$.*

Proof. Let $P = x \otimes f$ be a rank one idempotent. By Lemma 3.6, if $AP + PA$ does not have two distinct nonzero eigenvalues, then $\langle A^2x, f\rangle = 0$ or $\langle A^2x, f\rangle = \langle Ax, f\rangle^2$. Let $\varepsilon > 0$ be a small positive number. Assume $\langle A^2x, f\rangle = 0$. If $A^2x \neq 0$, take $h \in X^*$ such that $\langle A^2x, h\rangle \neq 0$ and let $P_\varepsilon = (1 + \varepsilon\langle x, h\rangle)^{-1} x \otimes (f + \varepsilon h)$; if $A^2x = 0$ and there exists $u \in X$ such that $\langle A^2u, f\rangle \neq 0$, let $P_\varepsilon = (1 + \varepsilon\langle u, f\rangle)^{-1}(x + \varepsilon u) \otimes f$; if $A^2x = 0$ and there exists no $u \in X$ such that $\langle A^2u, f\rangle \neq 0$, take u and h such that $\langle A^2u, h\rangle \neq 0$ and let $P_\varepsilon = \langle x + \varepsilon u, f + \varepsilon h\rangle^{-1}(x + \varepsilon u) \otimes (f + \varepsilon h)$. If $\langle A^2x, f\rangle = \langle Ax, f\rangle^2 \neq 0$, take any u so that $\{x, u\}$ is linearly independent and $\langle Au, f\rangle \neq 0$, and let $P_\varepsilon = (1 + \varepsilon\langle u, f\rangle)^{-1}(x + \varepsilon u) \otimes f$. In any case, for sufficient small ε, the rank one idempotent $P_\varepsilon = x_\varepsilon \otimes f_\varepsilon$ satisfies that $\langle A^2x_\varepsilon, f_\varepsilon\rangle \neq 0$, $\langle A^2x_\varepsilon, f_\varepsilon\rangle \neq \langle Ax_\varepsilon, f_\varepsilon\rangle^2$, $\lim_{\varepsilon\to 0} \|x_\varepsilon - x\| = 0$ and $\lim_{\varepsilon\to 0} \|f_\varepsilon - f\| = 0$.

For given A_i, $i = 1, 2, 3$, in the lemma, and for any given positive number $\delta > 0$, by Lemma 3.6, we have to show that for any rank one idempotent P there exists a rank one idempotent $Q = u \otimes h$ with $\|P - Q\| < \delta$ such that $\langle A_i^2u, h\rangle \neq 0$ and $\langle A_i^2u, h\rangle \neq \langle A_iu, h\rangle^2$, $i = 1, 2, 3$.

Given $\delta > 0$. If the rank one operator $P = x \otimes f$ is such that $\langle A_1^2x, f\rangle = 0$ or $\langle A_1^2x, f\rangle = \langle A_1x, f\rangle^2$, then, by what has been proved in the previous paragraph, there exists a rank one idempotent $Q_1 = u_1 \otimes h_1$ such that $\|P - Q_1\| < \frac{1}{3}\delta$, $\langle A_1^2u_1, h_1\rangle \neq 0$ and $\langle A_1^2u_1, h_1\rangle \neq \langle A_1u_1, h_1\rangle^2$. If both $\langle A_i^2u_1, h_1\rangle \neq 0$ and $\langle A_i^2u_1, h_1\rangle \neq \langle A_iu_1, h_1\rangle^2$ hold for $i = 2, 3$, then we are done. If, say, $\langle A_2^2u_1, h_1\rangle = 0$ or $\langle A_2^2u_1, h_1\rangle = \langle A_2u_1, h_1\rangle^2$, there exists a rank one idempotent $Q_2 = u_2 \otimes h_2$ with

$$\|u_1 - u_2\| < \max\left\{\frac{\delta}{6\|h_1\|}, \frac{1}{4\|A\|^2\|h_1\|}|\langle A^2u_1, h_1\rangle|\right\},$$

and

$$\|h_1 - h_2\| < \max\left\{\frac{\delta}{6(\|u_1\| + 1)}, \frac{1}{4\|A\|^2(\|u_1\| + 1)}|\langle A^2u_1, h_1\rangle|\right\}$$

such that $\langle A_2^2u_2, h_2\rangle \neq 0$ and $\langle A_2^2u_2, h_2\rangle \neq \langle A_2u_2, h_2\rangle^2$. Then $\|Q_1 - Q_2\| < \frac{1}{3}\delta$, $\langle A_1^2u_2, h_2\rangle \neq 0$, and $\langle A_1^2u_2, h_2\rangle \neq \langle A_1u_2, h_2\rangle^2$. If $\langle A_3^2u_2, h_2\rangle \neq 0$ and $\langle A_3^2u_2, h_2\rangle \neq \langle A_3u_2, h_2\rangle^2$, then we are done since $\|P - Q_2\| < \frac{2}{3}\delta$; if $\langle A_3^2u_2, h_2\rangle = 0$ or $\langle A_3^2u_2, h_2\rangle = \langle A_3u_2, h_2\rangle^2$, one may repeat the above process and find $Q_3 = u_3 \otimes h_3$ such that $\|Q_2 - Q_3\| < \frac{1}{3}\delta$, $\langle A_i^2u_3, h_3\rangle \neq 0$ and $\langle A_i^2u_3, h_3\rangle \neq \langle A_iu_3, h_3\rangle^2$ for all $i = 1, 2, 3$. Consequently, we get the desired $Q = Q_3$ as $\|P - Q_3\| < \delta$. ∎

Lemma 3.8. *Let X be a Banach space of dimension at least 2. Let P, Q in $\mathcal{I}_1(X)$ be such that $\sigma(PQ + QP) = \{0\}$. Then $PQ = 0 = QP$ if and only if there does not exist R in $\mathcal{I}_1(X)$ such that $(PR + RP)/2, (QR + RQ)/2 \in \mathcal{I}_1(X)$.*

Proof. Let $P, Q \in \mathcal{I}_1(X)$ such that $PQ = 0 = QP$. Then there is a decomposition of X so that P and Q have operator matrices

$$\mathrm{diag}\,(1, 0) \oplus 0 \qquad \text{and} \qquad \mathrm{diag}\,(0, 1) \oplus 0.$$

Then for any $R \in \mathcal{I}_1(X)$ such that $(PR + RP)/2 \in \mathcal{I}_1(X)$, the $(1, 1)$ entry of the operator matrix of R equals 1, and the off-diagonal part of the first row or the first column of the operator matrix of R must be zero to ensure that $PR + RP$ has rank one. Hence, R has operator matrix

$$\begin{pmatrix} 1 & * & * \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{pmatrix}.$$

Similarly, if $(QR + RQ)/2 \in \mathcal{I}_1(X)$, then R has operator matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ * & 1 & * \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & * & 0 \\ 0 & 1 & 0 \\ 0 & * & 0 \end{pmatrix}.$$

Thus, we cannot have $R \in \mathcal{I}_1(X)$ such that both $(PR + RP)/2, (QR + RQ)/2 \in \mathcal{I}_1(X)$.

Conversely, suppose $P, Q \in \mathcal{I}_1(X)$ are such that $\sigma(PQ + QP) = \{0\}$. If $PQ \neq 0$ or $QP \neq 0$, then there is a decomposition of X so that P has operator matrix $\mathrm{diag}\,(1, 0) \oplus 0$ and Q has operator matrix

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let R have operator matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $PR + RP$ has operator matrix

$$\begin{pmatrix} 2 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and $QR + RQ$ has operator matrix

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, $(PR + RP)/2, (QR + RQ)/2 \in \mathcal{I}_1(X)$. ■

For a Banach space X and a ring automorphism τ of $\mathbb{C}$, if an additive map $T : X \to X$ satisfies $T(\lambda x) = \tau(\lambda)Tx$ for all complex λ and all vectors x, we say that T is τ-linear. The following result can be proved by a similar argument as the proof of the main result in [18]; see also [5, Lemma 3] and [19, Theorem 2.3, 2.4].

Lemma 3.9. *Let X and Y be complex Banach spaces with dimension at least 3. Let $\Phi : \mathcal{I}_1(X) \to \mathcal{I}_1(Y)$ be a bijective map with the property that*

$$PQ = QP = 0 \quad \text{if and only if} \quad \Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$$

for all P, Q in $\mathcal{I}_1(X)$. Then there exists a ring automorphism τ of $\mathbb{C}$ such that one of the following cases holds.

(i) There exists a τ-linear transformation $T : X \to Y$ satisfying

$$\Phi(P) = TPT^{-1} \qquad \text{for all } P \in \mathcal{I}_1(X).$$

(ii) There exists a τ-linear transformation $T : X^* \to Y$ satisfying

$$\Phi(P) = TP^*T^{-1} \qquad \text{for all } P \in \mathcal{I}_1(X).$$

If X is infinite dimensional, the transformation T is an invertible bounded linear or conjugate linear operator.

We are now ready to complete the proof of Theorem 3.1. Recall that $s > r \geq 0$ and $m = r + s + 1 \geq 2$, and we assume from now on that X_2 has dimension at least 3.

Proof of Theorem 3.1. Recall that Φ satisfies condition (3.1).

Claim 1. *Φ is injective, and $\Phi(0) = 0$.*

It is just Corollary 3.4.

Claim 2. *If $A \in \mathcal{A}_1$ is a nonzero multiple of a rank one idempotent, then so is $\Phi(A)$. In particular, if $P \in \mathcal{I}_1(X_1)$, then $\Phi(P) = \mu R$ such that $R \in \mathcal{I}_1(X_2)$ and $\mu^m = 1$. When*

$s > r > 0$, the map Φ also sends square zero rank one operators to square zero rank one operators.

Let $A \neq 0$ be a nonzero multiple of an idempotent, say $A = \alpha P$, where $0 \neq \alpha \in \mathbb{C}$ and P in $\mathcal{A}_1$ is a rank one idempotent operator. For any D in $\mathcal{A}_2$ of rank at most 3, there is C in $\mathcal{A}_1$ such that $\Phi(C) = D$. By equation (3.1) we have

$$\sigma(D^r\Phi(A)D^s + D^s\Phi(A)D^r) = \sigma(C^rAC^s + C^sAC^r),$$

which contains 0 and has at most 2 nonzero elements. Putting $B = A$ in equation (3.1), we have $\sigma(2\Phi(A)^m) = \sigma(2A^m) \neq \{0\}$. Applying Lemma 2.1 or Corollary 2.3, depending on $s > r > 0$, or $s > r = 0$, we see that $\Phi(A)$ is a nonzero multiple of rank one idempotent. Thus Φ preserves nonzero multiples of rank one idempotents. If P in $\mathcal{A}_1$ is a rank one idempotent, then $\Phi(P) = \mu R$, where R in $\mathcal{A}_2$ is rank one idempotent and $\mu \in \mathbb{C}$. Since $\sigma(2P^m) = \sigma(2\Phi(P)^m)$, we see that $\mu^m = 1$. The last assertion follows from Lemma 2.1 and (3.1).

Suppose that $s > r > 0$. In this case, Φ sends rank one operators to rank one operators by Claim 2. Observe that if $\Phi(x \otimes f) = y \otimes g$, by (3.1) we will have

$$\left\langle \Phi(B)^{r+s}y, g\right\rangle = \left\langle B^{r+s}x, f\right\rangle \tag{3.3}$$

$$\langle y, g\rangle^{r+s-1}\langle \Phi(B)y, g\rangle = \langle x, f\rangle^{r+s-1}\langle Bx, f\rangle, \quad \forall B \in \mathcal{A}_1. \tag{3.4}$$

Setting $A = B = x \otimes f$, we also have

$$\langle y, g\rangle^{r+s+1} = \langle x, f\rangle^{r+s+1}. \tag{3.5}$$

With these three conditions (3.3), (3.4) and (3.5) in hand, we can now utilize the proof of [12, Theorem 2.5] to arrive at the desired assertions of Theorem 3.1.

Conclusion I. *From now on, we know that the case $s > r > 0$ is done.*

However, since we shall use some arguments below in the next section, the case $s > r > 0$ is still considered until we reach Conclusion II in the following.

Claim 3. *$\Phi(\alpha A) = \alpha\Phi(A)$ holds for all A in $\mathcal{I}_1(X_1)$ and α in $\mathbb{C}$.*

Denote $\Phi(A) = C$. Then, for any $B \in \mathcal{A}_1$, we have

$$\begin{aligned}
&\sigma(\Phi(B)^r\Phi(\alpha A)\Phi(B)^s + \Phi(B)^s\Phi(\alpha A)\Phi(B)^r)\\
= \;&\sigma(B^r(\alpha A)B^s + B^s(\alpha A)B^r)\\
= \;&\sigma(\alpha\Phi(B)^r\Phi(A)\Phi(B)^s + \alpha\Phi(B)^s\Phi(A)\Phi(B)^r)\\
= \;&\sigma(\Phi(B)^r(\alpha C)\Phi(B)^s + \Phi(B)^s(\alpha C)\Phi(B)^r)).
\end{aligned}$$

Since $\Phi(\mathcal{A}_1)$ contains $\mathcal{I}_1(X_2)$, Lemma 3.3 implies $\Phi(\alpha A) = \alpha C = \alpha\Phi(A)$.

Claim 4. *Suppose $\Phi(A)$ is a rank one idempotent. Then $A^2 \neq 0$.*

In the case $s > r > 0$, it follows from Lemma 2.1 and (3.1) that A has rank 1. Then by (3.1) again, A could not have zero trace. Thus $A^2 \neq 0$.

Next, we shall see that it is impossible to have $A^2 = 0$ when $s > r = 0$, either. Assuming $A^2 = 0$ and noting that $A \neq 0$, we would have a nonzero x in X_1 such that $\{x, Ax\}$ is linearly independent. Let $B = x \otimes f$ be any rank one idempotent on X_1 with $\langle Ax, f\rangle = 1$, and thus $\lambda\Phi(B) = y \otimes g \in \mathcal{I}_1(X_2)$ is a rank one idempotent on X_2 with some scalar λ such that λ^m=1. If $AB + BA$ is of rank 1, then either $\{x, Ax\}$ is linearly dependent or $\{f, A^* f\}$ is linearly dependent. However, $A^2 = 0$ would then establish a contradiction $x = 0$ or $f = 0$. On the other hand, as its trace $2\langle Ax, f\rangle = 2$, the Jordan product $AB + BA$ has exactly rank 2. By Lemma 3.6(2), we see that $AB + BA$ cannot have two distinct nonzero eigenvalues. This forces

$$\sigma(AB + BA) \cup \{0\} = \{0, 1\} = \sigma\left(\Phi(A)\Phi(B) + \Phi(A)\Phi(B)\right) \cup \{0\}. \tag{3.6}$$

As $\Phi(A)$ is a rank one idempotent, Lemma 3.6(1) implies that $\Phi(A)\Phi(B)+\Phi(A)\Phi(B)$ cannot have a nonzero repeated eigenvalue. Therefore, $\Phi(A)\Phi(B) + \Phi(A)\Phi(B)$ has rank 1. Consequently, $\{y, \Phi(A)y\}$ or $\{g, \Phi(A)^* g\}$ is linearly dependent. Since $\Phi(A)$ is an idempotent, we have exactly $y = \Phi(A)y$ or $g = \Phi(A)^* g$. Computing trace in (3.6), we have the absurd equality $1 = 2\lambda\langle y, g\rangle = 2\lambda$ with $\lambda^m = 1$.

Claim 5. *Let $\Phi(C) = \Phi(A) + \Phi(B)$. If $rs \neq 0$, then $C = A + B$. If $rs = 0$, then together with $A^2 \neq 0$, $B^2 \neq 0$ and $C^2 \neq 0$, it implies $C = A + B$.*

Let $W = \Phi(A)$ and $W' = \Phi(B)$. For any rank one idempotent $P \in \mathcal{A}_1$, by Claim 2, $Q = \lambda\Phi(P)$ is a rank one idempotent for some scalar λ with $\lambda^m = 1$. It follows from (3.1) that

$$\sigma(\lambda(Q^r(W + W')Q^s + Q^s(W + W')Q^r)) = \sigma(P^r CP^s + P^s CP^r),$$

$$\sigma(\lambda(Q^r WQ^s + Q^s WQ^r)) = \sigma(P^r AP^s + P^s AP^r),$$

and

$$\sigma(\lambda(Q^r W'Q^s + Q^s W'Q^r)) = \sigma(P^r BP^s + P^s BP^r).$$

If $rs \neq 0$, then the traces of the operators in each side of above equations are the same. This leads to

$$\operatorname{tr}(PCP) = \operatorname{tr}(\lambda Q(W + W')Q) = \operatorname{tr}(P(A + B)P)$$

for all rank one idempotents P in $\mathcal{A}_1$. Hence we have $C = A + B$ by Lemma 3.3.

Assume $rs = 0$. Then, for those rank one idempotent operators $P \in \mathcal{A}_1$ such that every one of $CP + PC$, $AP + PA$ and $BP + PB$ has two distinct nonzero eigenvalues, applying (3.1) and then taking trace, we have

$$\operatorname{tr}(PC) = \operatorname{tr}(P(A + B)). \tag{3.7}$$

By assumption, A, B and C are non square-zero. Lemma 3.7 ensures that (3.7) holds for a dense set of rank one idempotents P in $\mathcal{A}_1$. As a result, $C = A + B$.

Claim 6. *There exists a scalar λ with $\lambda^m = 1$ such that $\lambda^{-1}\Phi$ sends rank one idempotents to rank one idempotents.*

Let f be nonzero in X_1^*. Assume $\langle x_1, f\rangle = \langle x_2, f\rangle = 0$, and $\Phi(x_1 \otimes f) = \lambda_1 P_1$, $\Phi(x_2 \otimes f) = \lambda_2 P_2$, and $\Phi((\frac{x_1+x_2}{2}) \otimes f) = \lambda_3 P_3$ for some rank one idempotents P_1, P_2, P_3 and scalars $\lambda_1, \lambda_2, \lambda_3$ with $\lambda_1^m = \lambda_2^m = \lambda_3^m = 1$. By Claims 3, 4 and 5, we have

$$2\lambda_3 P_3 = \lambda_1 P_1 + \lambda_2 P_2.$$

Comparing traces, we have

$$2\lambda_3 = \lambda_1 + \lambda_2.$$

Since $\lambda_1^m = \lambda_2^m = \lambda^m = 1$, we have

$$\lambda_1 = \lambda_2 = \lambda_3.$$

Denote this common value by λ_f. Similarly, for any nonzero x in X_1 we will have an mth root λ_x of unity depending only on x such that

$$\Phi(x \otimes f) = \lambda_x Q_{x\otimes f}$$

for some rank one idempotent $Q_{x\otimes f}$ whenever $f(x) = 1$.

Now consider any two rank one idempotents $x_1 \otimes f_1$ and $x_2 \otimes f_2$ in $\mathcal{A}_1$. We write $x_1 \otimes f_1 \sim x_2 \otimes f_2$ if there is a scalar λ with $\lambda^m = 1$ such that $\lambda\Phi(x_i \otimes f_i)$ is a rank one idempotent for $i = 1, 2$. In case $\alpha = \langle x_1, f_2\rangle \neq 0$, we see that

$$x_1 \otimes f_1 \sim x_1 \otimes \frac{f_2}{\alpha} = \frac{x_1}{\alpha} \otimes f_2 \sim x_2 \otimes f_2.$$

In case $\langle x_1, f_2\rangle = \langle x_2, f_1\rangle = 0$, we also have

$$x_1 \otimes f_1 \sim (x_1 + x_2) \otimes f_1 \sim (x_1 + x_2) \otimes f_2 \sim x_2 \otimes f_2.$$

Conclusion II. *By Claim 6, without loss of generality, we assume that Φ preserves rank one idempotents. By Conclusion I, it suffices to deal with the case $s > r = 0$ in the sequel .*

Claim 7. *If $\Phi(A) \in \mathcal{A}_2$ is a rank one idempotent, then $A \in \mathcal{A}_1$ is a rank one idempotent.*

Suppose $\Phi(A)$ is a rank one idempotent. If A is of rank one, then Claims 1 and 3 ensure that A is a rank one idempotent. Now we suppose A has rank at least 2, and we want to derive a contradiction. Note that $A^2 \neq 0$ by Claim 4.

Case 1. Suppose there is an x in X_1 such that $\{x, Ax, A^2x\}$ is linearly independent. Let f in X_1^* be such that $\langle x, f\rangle = \langle Ax, f\rangle = 1$, but $\langle A^2x, f\rangle \neq 0$ or 1. Lemma 3.6(2) ensures that $A(x \otimes f) + (x \otimes f)A$ has 2 distinct nonzero eigenvalues, and so has $\Phi(A)(y \otimes g) + (y \otimes g)\Phi(A)$ by (3.1), where $y \otimes g = \Phi(x \otimes f)$ is a rank one idempotent. Comparing traces, we have $\langle \Phi(A)y, g\rangle = \langle Ax, f\rangle = 1$. This contradicts to Lemma 3.6(2), however.

Case 2. Suppose $\{x, Ax, A^2x\}$ is linearly dependent for all x in X_1. Hence, by Kaplansky's Lemma ([15, 1]) there are scalars a, b, c, not all zero, such that $aA^2 + bA + cI = 0$.

Subcase 2a. If A has rank 2 then A has nonzero eigenvalues α_1, α_2 (maybe equal). With respect to a suitable space decomposition, we can assume

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & \alpha_2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \alpha_1 & 1 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then

$$A = \alpha_1 e_1 \otimes e_1 + \alpha_2 e_2 \otimes e_2 \quad \text{or} \quad A = \alpha_1 e_1 \otimes e_1 + \alpha_1 (\frac{e_1}{\alpha_1} + e_2) \otimes e_2.$$

By Claims 3 and 5, and Conclusion II, the rank one idempotent

$$\begin{aligned} \Phi(A) &= \Phi(\alpha_1 e_1 \otimes e_1 + \alpha_2 e_2 \otimes e_2) \\ &= \alpha_1 \Phi(e_1 \otimes e_1) + \alpha_2 \Phi(e_2 \otimes e_2) \\ &= \alpha_1 y_1 \otimes g_1 + \alpha_2 y_2 \otimes g_2, \end{aligned}$$

in the first case with rank one idempotents $y_1 \otimes g_1 = \Phi(e_1 \otimes e_1)$ and $y_2 \otimes g_2 = \Phi(e_2 \otimes e_2)$. Observing ranks, we see that $\{y_1, y_2\}$ or $\{g_1, g_2\}$ is linearly dependent. On the other hand, as $\langle e_1, e_2 \rangle \langle e_2, e_1 \rangle = 0$ we see by (3.1) that $\langle y_2, g_1 \rangle \langle y_1, g_2 \rangle = 0$. This eventually gives the contradiction $1 = \langle y_1, g_1 \rangle \langle y_2, g_2 \rangle = 0$. The second case is similar.

Subcase 2b. Assume A has rank at least 3. Since A is quadratic, each Jordan block of A has order either 1 or 2. Consider the case

$$A = \begin{pmatrix} \alpha_1 & 1 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & * \end{pmatrix}.$$

Here the nonzero eigenvalues α_1, α_2 of A can be equal. Then

$$Ae_1 = \alpha e_1, \quad Ae_2 = e_1 + \alpha_1 e_2 \quad \text{and} \quad Ae_3 = \alpha_2 e_3.$$

Observe

$$\begin{aligned} A(e_1 \otimes e_1) + (e_1 \otimes e_1)A &= e_1 \otimes (2\alpha_1 e_1 + e_2), \\ A(e_2 \otimes e_2) + (e_2 \otimes e_2)A &= (e_1 + 2\alpha_1 e_2) \otimes e_2, \end{aligned}$$

and

$$A(e_3 \otimes e_3) + (e_3 \otimes e_3)A = 2\alpha_2 e_3 \otimes e_3.$$

Consider the rank one idempotents $\Phi(A) = y \otimes g$, and $\Phi(e_i \otimes e_i) = y_i \otimes g_i$ for $i = 1, 2, 3$. By (3.1), we see that

$$\sigma((y \otimes g)(y_i \otimes g_i) + (y_i \otimes g_i)(y \otimes g)) \cup \{0\} = \{0, 2\alpha_1\} \text{ or } \{0, 2\alpha_2\}, \quad \text{for } i = 1, 2, 3.$$

In particular, by Lemma 3.6(1),

$$\langle y_i, g\rangle \langle y, g_i\rangle = \alpha_1 \text{ or } \alpha_2, \text{ is not zero, for } i = 1, 2, 3. \tag{3.8}$$

But as $\langle e_i, e_j\rangle \langle e_j, e_i\rangle = 0$, we have

$$\langle y_i, g_j\rangle \langle y_j, g_i\rangle = 0 \quad \text{whenever } i \neq j.$$

On the other hand, Lemma 3.6(1) and (3.8) force all $(y\otimes g)(y_i\otimes g_i)+(y_i\otimes g_i)(y\otimes g)$ have rank one. Consequently, $\{y_i, y\}$ or $\{g, g_i\}$ is linearly dependent for each $i = 1, 2, 3$. Eventually, we might have two of y_1, y_2, y_3 are linearly dependent, or two of g_1, g_2, g_3 are linearly dependent. Suppose y_1, y_2 are dependent. Since $g_1(y_1) = g_2(y_2) = 1$, we see that $\langle y_1, g_2\rangle \langle y_2, g_1\rangle = 0$, which is absurd. We shall reach other contradictions similarly for other possible situations. Analogously, we can also derive a contradiction when we are dealing with the case

$$A = \begin{pmatrix} \alpha_1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 \\ 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & * \end{pmatrix} \quad \text{or} \quad A = \begin{pmatrix} \alpha_1 & 1 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & 0 & 0 \\ 0 & 0 & \alpha_2 & 1 & 0 \\ 0 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & 0 & 0 & * \end{pmatrix}.$$

This completes the verification of Claim 7.

Claim 8. *One of the following statements is true.*

(i) *There exists a bounded invertible linear operator $T : X_1 \to X_2$ such that*

$$\Phi(x \otimes f) = T(x \otimes f)T^{-1} \quad \text{for all } x \in X_1, f \in X_1^* \text{ with } \langle x, f\rangle = 1.$$

(ii) *There exists a bounded invertible linear operator $T : X_1^* \to X_2$ such that*

$$\Phi(x \otimes f) = T(x \otimes f)^*T^{-1} \quad \text{for all } x \in X_1, f \in X_1^* \text{ with } \langle x, f\rangle = 1.$$

Since Φ preserves rank one idempotents in both directions, by use of Lemma 3.8, it is easily checked that $P, Q \in \mathcal{I}_1(X)$ satisfy $PQ = 0 = QP$ if and only if $\Phi(P)\Phi(Q) = 0 = \Phi(Q)\Phi(P)$. Thus we can apply Lemma 3.9 to conclude that (i) or (ii) holds, but with T a τ-linear for some ring automorphism τ of $\mathbb{C}$.

Next we prove that τ is the identity and hence T is linear. For any $\alpha \in \mathbb{C} \setminus \{1, 0\}$, let A and B have operator matrices

$$\begin{pmatrix} 1 & \alpha - 1 \\ 0 & 0 \end{pmatrix} \oplus 0 \qquad \text{and} \qquad \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \oplus 0.$$

Then $AB + BA$ has two distinct nonzero eigenvalues summing up to 2α. Since

$$\begin{aligned} & \sigma(AB + BA) = \sigma(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) \\ = \;& \sigma(T(AB + BA)T^{-1}) = \{\tau(\xi) : \xi \in \sigma(AB + BA)\}, \end{aligned}$$

we see that

$$2\alpha = \mathrm{tr}\,(AB + BA) = \mathrm{tr}\,(\Phi(A)\Phi(B) + \Phi(B)\Phi(A)) = \mathrm{tr}\,(T(AB + BA)T^{-1}) = 2\tau(\alpha).$$

Hence $\tau(\alpha) = \alpha$ for any $\alpha \in \mathbb{C}$. It follows that T is an invertible bounded linear operator.

Claim 9. *Φ has the form in Theorem 3.1.*

Suppose (i) in Claim 8 holds. Let $A \in \mathcal{A}_1$ be arbitrary. For any $x \in X_1$ and $f \in X_1^*$ with $\langle x, f\rangle = 1$, the condition (3.1) ensures that

$$\begin{aligned} & \sigma((T^{-1}\Phi(A)T)(x \otimes f)^s + (x \otimes f)^s(T^{-1}\Phi(A)T)) \\ = \;& \sigma(T[T^{-1}\Phi(A)T(x \otimes f)^s + (x \otimes f)^s T^{-1}\Phi(A)T]T^{-1}) \\ = \;& \sigma(A(x \otimes f)^s + (x \otimes f)^s A). \end{aligned}$$

Hence, by Lemma 3.3, we have

$$\Phi(A) = TAT^{-1}$$

for all A in $\mathcal{A}_1$, that is, Φ has the form (1) in the theorem.

Similarly, one can show that Φ has the form (2) if (ii) of Claim 8 holds. ■

4. Generalized Jordan product spectrum preserving maps of self-adjoint operators

Let H be a complex Hilbert space and $\mathcal{S}(H)$ be the real linear space of all self-adjoint operators in $\mathcal{B}(H)$. Note that $\mathcal{S}(H)$ is a Jordan algebra. In this section we solve the problems discussed previously for maps on $\mathcal{S}(H)$. Our results refine those in [7].

Theorem 4.1. *For $i = 1, 2$, let H_i be a complex Hilbert space, and $\mathcal{S}(H_i)$ be the Jordan algebra of all bounded self-adjoint operators on H_i. Consider the product $T_1 \circ \cdots \circ T_k$ defined in Definition 1.1. Suppose $\Phi : \mathcal{S}(H_1) \to \mathcal{S}(H_2)$ satisfies*

$$\sigma(\Phi(A_1) \circ \Phi(A_2) \circ \cdots \circ \Phi(A_k)) = \sigma(A_1 \circ A_2 \circ \cdots \circ A_k), \tag{4.1}$$

whenever any one of the A_i's has rank at most one. Suppose further that the range of ϕ contains all self-adjoint operators of rank at most 3. Then there exist a scalar ξ in $\{-1, 1\}$ with $\xi^m = 1$ and a unitary operator $U : H_1 \to H_2$ such that either

$$\Phi(A) = \xi UAU^* \quad \text{for all } A \text{ in } \mathcal{S}(H_1),$$

or

$$\Phi(A) = \xi UA^tU^* \quad \text{for all } A \text{ in } \mathcal{S}(H_1),$$

where A^t is the transpose of A for an arbitrarily but fixed orthonormal basis.

To prove Theorem 4.1, it is important to characterize rank one operators in terms of the general Jordan products of self-adjoint operators. We have the following lemma.

Lemma 4.2. *Suppose $s > r \geq 0$ is a pair of nonnegative integers. Let H be a Hilbert space of dimension at least three, and let $0 \neq A \in \mathcal{S}(H)$. Then the following statements are equivalent.*

(a) A has rank one.

(b) For any $B \in \mathcal{S}(H)$, $\sigma(B^rAB^s + B^sAB^r)$ contains 0 and at most two nonzero elements.

(c) There does not exist $B \in \mathcal{S}(H)$ of rank at most three such that $B^rAB^s + B^sAB^r$ has rank at most three and $\sigma(B^rAB^s + B^sAB^r)$ contains three distinct nonzero elements.

Proof. The implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) are clear. To prove (c) $\Rightarrow$ (a), we consider the contrapositive. Suppose (a) does not hold. Assume $rs \neq 0$. If A has rank at least 3, then there are vectors x_1, x_2, x_3 such that $\{Ax_1, Ax_2, Ax_3\}$ is linearly independent. Extend an orthonormal basis for $[x_1, x_2, x_3, Ax_1, Ax_2, Ax_3]$ to an orthonormal basis for H. Then the operator matrix of A with respect to this basis has the form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{pmatrix},$$

where $A_{11} = A_{11}^*$ is the compression of A on the subspace $[x_1, x_2, x_3, Ax_1, Ax_2, Ax_3]$. By [12, Lemma 2.3], we can choose an orthonormal basis for $[x_1, x_2, x_3, Ax_1, Ax_2, Ax_3]$ so that the leading 3×3 matrix of A_{11} equals $\operatorname{diag}(a_1, a_2, a_3)$ for some nonzero scalars a_1, a_2, a_3. Now construct B so that the operator matrix of B using the same basis as that of A equals $\operatorname{diag}(1, b_2, b_3) \oplus 0 \oplus 0$ so that $a_1, a_2b_2^{r+s}, a_3b_3^{r+s}$ are distinct nonzero numbers. Then $B^rAB^s + B^sAB^r$ has rank 3 with three distinct nonzero eigenvalues.

Next, suppose A has rank 2. Choosing a suitable basis, we may assume that A has operator matrix $\operatorname{diag}(a, b, 0) \oplus 0$. Construct B with operator matrix $[d] \oplus B_1 \oplus 0$, where

$$B_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2 \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}.$$

Compute

$$B_1^k = 2^{k-1} \left[2^k \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \right], \qquad k = 1, 2, \ldots.$$

Now, if $\gamma = 2^r$ and $\delta = 2^s$ then

$$B_1^r \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1^s + B_1^s \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} B_1^r = 2^{r+s-1} \begin{pmatrix} (\gamma+1)(\delta+1) & \gamma\delta - 1 \\ \gamma\delta - 1 & (\gamma-1)(\delta-1) \end{pmatrix}$$

has determinant $-4^{r+s-1}(\gamma - \delta)^2 < 0$. So, it has a positive and a negative eigenvalue, say, μ and ν. Thus, we can choose d so that $B^rAB^s + B^sAB^r$ has three nonzero distinct nonzero eigenvalues: $2ad^{r+s}, b\mu, b\nu$.

Next, suppose $s > r = 0$. If A has rank 2, then A has an operator matrix of the form $\operatorname{diag}(a_1, a_2, 0) \oplus 0$ for some nonzero real numbers a_1, a_2. Let $b > 0$ be such that $2b^sa_1 \neq$

$a_2(1/2 \pm 1/\sqrt{2})$. Suppose $B \in \mathcal{S}(H)$ is such that B and $AB^s + B^sA$ have operator matrices

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \oplus 0 \qquad \text{and} \qquad \begin{pmatrix} 2a_1b^s & 0 & 0 \\ 0 & a_2 & a_2/2 \\ 0 & a_2/2 & 0 \end{pmatrix} \oplus 0.$$

Then $AB^s + B^sA$ has rank 3 with three distinct nonzero eigenvalues $2b^sa_1, a_2(1/2 + 1/\sqrt{2})$ and $a_2(1/2 - 1/\sqrt{2})$.

Now, suppose A has rank at least 3. If $A = \lambda I$, then let B have operator matrix $\mathrm{diag}\,(1, 2, 3) \oplus 0$ with respect to some orthonormal basis for H. Then B has rank 3 and $AB^s + B^sA$ has rank 3 with three distinct nonzero eigenvalues $\lambda, 2^s\lambda, 3^s\lambda$. So, assume A is non-scalar. Thus, there is a unit vector $x_1 \in H$ such that $Ax_1 = a_1x_1 + a_2x_2$ with $a_1 \neq 0$ and $a_2 > 0$, where x_2 is a unit vector in $[x_1]^\perp$. Let $Ax_2 = b_1x_1 + b_2x_2 + b_3x_3$ with $b_3 \geq 0$, where x_3 is a unit vector in $[x_1, x_2]^\perp$. We consider two cases.

Case 1. If $b_3 > 0$, then the operator matrix of the self-adjoint operator A with respect to an orthonormal basis with $\{x_1, x_2, x_3\}$ as the first three vectors has the form

$$\begin{pmatrix} a_1 & a_2 & 0 & 0 \\ a_2 & b_2 & b_3 & 0 \\ 0 & b_3 & * & * \\ 0 & 0 & * & * \end{pmatrix}.$$

Let B have operator matrix $I_2 \oplus 0$. Then $AB^s + B^sA$ has an operator matrix of the form $C_1 \oplus 0$, where

$$C_1 = \begin{pmatrix} 2a_1 & 2a_2 & 0 \\ 2a_2 & 2b_2 & b_3 \\ 0 & b_3 & 0 \end{pmatrix}.$$

Note that $\det(C_1) = -2a_1b_3^2 \neq 0$, and $C_1 - \lambda I$ has rank at least two for any eigenvalue λ as the 2×2 submatrix at the right top corner is always invertible. So, C_1 is invertible and has three distinct nonzero eigenvalues. Hence, $AB^s + B^sA$ has rank 3 with three distinct nonzero eigenvalues.

Case 2. Suppose $b_3 = 0$. Then $[x_1, x_2]$ is an invariant subspace of A. Since A has rank at least 3, there is a unit vector x_3 in H such that $Ax_3 \neq 0$ and $Ax_3 \in \{x_1, x_2\}^\perp$.

Subcase 2a. If $[x_1, x_2, x_3]$ is an invariant subspace of A, then with respect to an orthonormal basis for $[x_1, x_2, x_3]$ and its orthonormal complement, A has operator matrix $A_1 \oplus A_2$, where A_1 in M_3 has rank at least 2. If A_1 has rank 3, we may assume that $A_1 = \mathrm{diag}\,(a_1, a_2, a_3)$. We can choose B with operator matrix $\mathrm{diag}\,(b_1, b_2, b_3) \oplus 0$ for some suitable b_1, b_2, b_3 so that $AB^s + B^sA$ has rank 3 with three distinct nonzero eigenvalues $2a_1b_1^s, 2a_2b_2^s, 2a_3b_3^s$. If A_1 has rank 2, we may assume that $A_1 = \mathrm{diag}\,(a_1, a_2, 0)$ and continue

exactly as when A has rank 2. Then choose B with operator matrix

$$\begin{pmatrix} b & 0 & 0 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \oplus 0$$

so that $2b^s a_1 \neq a_2(1/2 \pm 1/\sqrt{2})$. Then $AB^s + B^s A$ has rank 3 with three distinct nonzero eigenvalues $2b^s a_1 \neq a_2(1/2 \pm 1/\sqrt{2})$.

SUBCASE 2B. Suppose $Ax_3 = c_3x_3 + c_4x_4$ so that $c_4 > 0$ and $\{x_1, x_2, x_3, x_4\}$ is an orthonormal set in H. If $Ax_4 = d_3x_3 + d_4x_4 + d_5x_5$ so that $\{x_3, x_4, x_5\}$ is an orthonormal set in H and $d_5 > 0$, then we are back to Case 1 with (x_1, x_2) replaced by (x_3, x_4). We thus assume that $[x_1, x_2, x_3, x_4]$ is an invariant subspace of A. With respect to an orthonormal basis for $[x_1, x_2, x_3, x_4]$ and its orthonormal complement, A has operator matrix $A_3 \oplus A_4$, where $A_3 \in M_4$ is self-adjoint and has rank at least 2. We may assume that A_3 is in diagonal form with at least two nonzero diagonal entries. Using a similar argument as in Subcase 2A, we get the desired conclusion. ■

Proof of Theorem 4.1. Assume that Φ satisfies (4.2). Let

$$r = \min\{p-1, m-p\} \quad \text{and} \quad s = \max\{p-1, m-p\}.$$

In particular, $r+s = m-1$. It suffices to prove a special case of Theorem 4.1, as that Theorem 3.1 to Theorem 1.2 in last section. More precisely, we assume the condition

$$\sigma(\Phi(B)^r\Phi(A)\Phi(B)^s + \Phi(B)^s\Phi(A)\Phi(B)^r) = \sigma(B^rAB^s + B^sAB^r) \tag{4.2}$$

holds whenever A or B in $\mathcal{S}(H_1)$ has rank at most one. The case $s = r$ has been done in [12]. Hence, we assume $s > r \geq 0$. Arguing similarly as in the beginning of the proof of Theorem 3.1, we can verify the case $\dim H_2 \leq 2$. Therefore, we assume the dimension of the Hilbert space H_2 is at least three in the sequel.

Claim 1. *Φ is injective, and $\Phi(0) = 0$.*

This works out similarly as in Corollary 3.4.

Claim 2. *Φ sends rank one self-adjoint operators to rank one self-adjoint operators.*

This follows from (4.2) and Lemma 4.2. Indeed, every rank one self-adjoint operator has the form $\pm x \otimes x$. So, $\Phi(x \otimes x) = \lambda_x y_x \otimes y_x$ for some $\lambda_x \in \{-1, 1\}$ and $y_x \in H_2$. Since

$$\{2\|x\|^{2m}, 0\} = \sigma(2(x \otimes x)^m) = \sigma(2\Phi(x \otimes x)^m) = \{2\lambda_x^m\|y_x\|^{2m}, 0\},$$

we see that λ_x is an mth root of the unity and $\|y_x\| = \|x\|$.

Claim 3. *Φ is real homogeneous; and if $\Phi(C) = \Phi(A)+\Phi(B)$ then $C = A+B$. Moreover, there is a fixed λ, being either $+1$ or -1, such that for every x in H_1 we have $\Phi(x \otimes x) = \lambda y_x \otimes y_x$ with $\|y_x\| = \|x\|$.*

The assertions follow from arguments similar to, and a bit easier than, that in Claims 3, 5 and 6 in the proof of Theorem 3.1 in last section.

Claim 4. *Φ has the form stated in the theorem.*

Let x, x' be two nonzero vectors in H_1, and $x \otimes x$ and $x' \otimes x'$ be the associated rank one self-adjoint operators, respectively. By (4.2), and Lemma 3.6 when $s > r = 0$, we see that

$$\operatorname{tr}(\Phi(x \otimes x)\Phi(x' \otimes x')) = \operatorname{tr}((x \otimes x)(x' \otimes x')),$$

or

$$\langle \lambda_x y_x, \lambda_{x'} y_{x'} \rangle = \langle x, x' \rangle .$$

This gives

$$|\langle y_x, y_{x'} \rangle| = |\langle x, x' \rangle|, \quad \text{for all nonzero } x, x' \in H_1.$$

If follows from the Wigner's Theorem [10] that there exist a modular one function $\xi : H_1 \to \mathbb{C}$ and a linear or conjugate linear isometry $U : H_1 \to H_2$ such that

$$y_x = \xi(x) U x, \quad \forall x \in H_1.$$

By Claim 3, we see that all $\xi(x)$ equal a constant $\xi \in \{-1, +1\}$, and

$$\Phi(x \otimes x) = \xi Ux \otimes Ux \quad \text{for all rank one projection } x \otimes x \text{ on } H_1.$$

Moreover, (4.2) ensures that $\xi^m = 1$. Because the range of Φ contains all rank one self-adjoint operators, by (4.2) we can see that U has dense range, and thus U is a unitary or a conjugate unitary operator.

In general, for any A in $\mathcal{S}(H_1)$, let $A_{i_p} = A$ and $A_{i_q} = x \otimes x$ with $\|x\| = 1$ if $q \neq p$, and substitute them into (4.2). Since both A and $\Phi(A)$ are self-adjoint, we see that

$$\begin{aligned} & \sigma(\xi^{m-1}((x \otimes x)^r U^* \Phi(A) U (x \otimes x)^s + (x \otimes x)^s U^* \Phi(A) U (x \otimes x)^r)) \\ = \; & \sigma((x \otimes x)^r A (x \otimes x)^s + (x \otimes x)^s A (x \otimes x)^r). \end{aligned}$$

By Lemma 3.6 and comparing traces, we get $\Phi(A) = \xi U A U^*$ for all A in $\mathcal{S}(H_1)$. If U is a conjugate unitary, take an orthonormal basis $\{e_j\}$ of H_1 and define a conjugate unitary $J : H_1 \to H_1$ by $J : \sum_j \xi_j e_j \mapsto \sum_j \bar{\xi}_j e_j$ and let $V = UJ$. Then V is unitary and $JA^*J = A^t$. Thus, $\Phi(A) = VA^tV^*$ for all A in $\mathcal{S}(H_1)$. ■

References

[1] B. Aupetit, "An improvement of Kaplansky's lemma on locally algebraic operators", Studia Math. 88 (1988), 275-278.

[2] B. Aupetit, Spectrum-preserving linear mappings between Banach algebras or Jordan-Banach algebras, J. London Math. Soc. (2) 62 (2000), 917-924.

[3] B. Aupetit and H. du Toit Mouton, Spectrum preserving linear mappings in Banach algebras, Studia Math. 109 (1994), 91-100.

[4] Z.-F. Bai and J.-C. Hou, Characterizing isomorphisms between standard operator algebras by spectral functions, J. Operator Theory 54 (2005), no. 2, 291–303.

[5] Z.-F. Bai and J.-C. Hou, Maps preserving zero-products or Jordan zero-products, preprint.
[6] M. Brešar and P. Šemrl, Invertibility preserving maps preserve idempotents, Michigan J. Math. 45 (1998), 483-488.
[7] J.T. Chan, C.K. Li and N.S. Sze, Mappings preserving spectra of product of matrices, Proc. Amer. Math. Soc. 135 (2007), no. 4, 977–986.
[8] J.-L. Cui and J.-C. Hou, Additive maps on standard operator algebras preserving parts of the spectrum, J. Math. Anal. Appl. 282 (2003), 266-278.
[9] J.-L. Cui and J.-C. Hou, Linear maps between Banach algebras compressing certain spectral functions, The Rocky Mountain Journal of Mathematics, 34:2 (2004), 565-584.
[10] M. Gyõry, A new proof of Wigner's theorem, Rep. Math. Phys., 54:2 (2004), 159-167.
[11] J. Hou, Rank preserving linear maps on $\mathcal{B}(X)$, Sci. China Ser. A 32 (1989), 929-940.
[12] J. Hou, C.K. Li and N.C. Wong, Jordan isomorphisms and maps preserving spectra of certain operator products, Studia Math. 184 (2008), no. 1, 31–47.
[13] L. Huang and J. Hou, Maps preserving spectral functions of operator products, Chinese Ann. Math. Ser. A 28 (2007), no. 6, 769–780.
[14] A.A. Jafarian and A.R. Sourour, Spectrum-preserving linear maps, J. Funct. Anal. 66 (1986), 255-261.
[15] I. Kaplansky, Infinite abelian groups, U. of Michigan Press, Ann Arbor, Michigan, 1954.
[16] I. Kaplansky, Algebraic and analytic aspects of operator algebras, CBMS Reg. Conf. Ser. in Math., vol. 1, Amer. Math. Soc., Providence, 1970.
[17] C.K. Li and N.K. Tsing, Linear preserver problems: A brief introduction and some special techniques, Linear Algebra Appl. 162-164 (1992), 217-235.
[18] L. Molnàr, Orthogonality preserving transformations on indefinite inner product space: Generalization of Uhlhorn's version of Wigner's theorem, J. Funct. Anal., 194 (2002), 248-262.
[19] P. Šemrl, Non-linear commutativity preserving maps, Acta Sci. Math. (Szeged) 71 (2005), no. 3-4, 781–819.
[20] A.R. Sourour, Invertibility preserving linear maps on $\mathcal{L}(X)$, Trans. Amer. Math. Soc. 348 (1996), 13-30.
[21] Q. Wang and J. Hou, Point-spectrum preserving elementary operators on $\mathcal{B}(H)$, Proc. Amer. Math. Soc. 126 (1998), 2083-2088.
[22] X. Zhang and J. Hou, Positive elementary operators compressing spectrum, Chinese Sci. Bull. 42 (1997), 270-273.

(Hou) Department of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, P. R. of China

E-mail address: houjinchuan@tyut.edu.cn

(Li) Department of Mathematics, The College of William & Mary, Williamsburg, VA 13185, USA. Li is an honorary professor of the University of Hong Kong

E-mail address: ckli@math.wm.edu

(Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw

ADDENDUM TO "MAPS PRESERVING THE SPECTRUM OF GENERALIZED JORDAN PRODUCT OF OPERATORS"

JINCHUAN HOU, CHI-KWONG LI, AND NGAI-CHING WONG

Regarding our paper [1], Jianlian Cui pointed out that some arguments in the proof of Theorem 3.1 are not entirely clear and accurate. Here we supply some details.

Theorem 3.1. ([1]) *Suppose a map $\Phi : \mathcal{A}_1 \to \mathcal{A}_2$ between standard operator algebras satisfies*

$$\sigma(\Phi(B)^r\Phi(A)\Phi(B)^s + \Phi(B)^s\Phi(A)\Phi(B)^r) = \sigma(B^rAB^s + B^sAB^r), \tag{3.1}$$

whenever A or B has rank at most one. Suppose also that the range of Φ contains all operators in $\mathcal{A}_2$ of rank at most 3. Then one of the following two assertions holds with $m = r + s + 1$.

(1) There exist a scalar λ with $\lambda^m = 1$ and an invertible operator T in $\mathcal{B}(X_1, X_2)$ such that

$$\Phi(A) = \lambda TAT^{-1} \quad \text{for all } A \text{ in } \mathcal{A}_1.$$

(2) The spaces X_1 and X_2 are reflexive, and there exist a scalar λ with $\lambda^m = 1$ and an invertible operator $T \in \mathcal{B}(X_1^*, X_2)$ such that

$$\Phi(A) = \lambda TA^*T^{-1} \quad \text{for all } A \text{ in } \mathcal{A}_1.$$

Some modifications in the proof of Theorem 3.1. First, remove the paragraph "In the case $s > r > 0, \ldots.$ Thus $A^2 \neq 0$" after **Claim 4**, as we do not need this in the proof.

Claim 6. *There exists a scalar λ with $\lambda^m = 1$ such that $\lambda^{-1}\Phi$ sends rank one idempotents to rank one idempotents.*

First line of the proof should be "Let f be nonzero in X_1^*. Assume $\langle x_1, f\rangle = \langle x_2, f\rangle = 1$". If $rs \neq 0$, the original proof works. In case $s > r = 0$ the proof can continue as follow. "By Claim 2, $\Phi(x_1 \otimes f) = \lambda_1 y_1 \otimes g_1$ and $\Phi(x_2 \otimes f) = \lambda_2 y_2 \otimes g_2$, where $g_1(y_1) = g_2(y_2) = 1$ and $\lambda_1^{s+1} = \lambda_2^{s+1} = 1$. Using the spectrum equation (3.1) we have

$$\begin{aligned}
&\sigma(\lambda_1^s\lambda_2(g_1(y_2)\, y_1 \otimes g_2 + g_2(y_1)\, y_2 \otimes g_1)) \\
&= \sigma(\lambda_2^s\lambda_1(g_2(y_1)\, y_2 \otimes g_1 + g_1(y_2)\, y_1 \otimes g_2)) \\
&= \sigma((x_1 \otimes f)(x_2 \otimes f) + (x_2 \otimes f)(x_1 \otimes f)) \\
&= \{0, 2\}.
\end{aligned}$$

By Lemma 3.6(1) and computing traces, we would have

$$\lambda_1^s\lambda_2 g_1(y_2)g_2(y_1) = \lambda_1\lambda_2^s g_1(y_2)g_2(y_1) = 1.$$

Date: April 20, 2010.

2002 Mathematical Subject Classification. Primary 47B49; 47A12.

Key words and phrases. Standard operator algebra, spectral functions, Jordan triple-products of operators, skew products of operators, nonlinear-preserving problems.

In particular, $\lambda_1^2 = \lambda_2^2$. Suppose $\lambda_1 = -\lambda_2$. Then we have $g_1(y_2)g_2(y_1) = -1$, and by Lemma 3.6(2), we will get a contradiction. So $\lambda_1 = \lambda_2$." At this point we can go back to the original proof again.

Claim 7. *If $\Phi(A) \in \mathcal{A}_2$ is a rank one idempotent, then $A \in \mathcal{A}_1$ is a rank one idempotent.*

Assume that $\Phi(A)$ is a rank one idempotent. Suppose A is not a rank one idempotent.

Case 1. As is.

Case 2. Suppose $\{x, Ax, A^2x\}$ is always linearly dependent. Then by Kaplansky's result, A is a quadratic operator, i.e., there is a, b such that $(A - aI)(A - bI) = 0$. Then with respect to a suitable space decomposition of X_1, A has an operator matrix of the form

$$\begin{pmatrix} aI & T \\ 0 & bI \end{pmatrix},$$

where T may be assumed to be zero if $a \neq b$.

If A has rank one, then A has operator matrix $[a] \oplus 0$. By Claim 3, we have $a = 1$ and we are done. So, assume that A has rank at least two. If a, b are distinct, we may assume that $a \neq 0$ and the null space of $A - aI$ has dimension at least 2 as X_1 has dimension at least 3. Moreover, we may assume that A has operator matrix $aI \oplus bI$. Suppose B_1, B_2, B_3 have operator matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0, \quad \text{and} \quad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \oplus 0,$$

we see that $\sigma(B_j^s A + AB_j^s) = \{2a, 0\}$ and $\Phi(B_j)$ is a rank one idempotent for $j = 1, 2, 3$.

Note that all $\Phi(A)$, $\Phi(B_1)$, $\Phi(B_2)$ and $\Phi(B_3)$ are rank one idempotents, and thus we can find a subspace V of X_2 of dimension at most 4 such that in a suitable space decomposition $X_2 = V \oplus V'$ these operators can be written as direct sums of 4×4 matrices and zero. So we might assume that X_2 has dimension at most 4 in the following discussion.

Assume that $\Phi(A)$ and $\Phi(B_1)$ have operator matrices

$$[1] \oplus 0 \quad \text{and} \quad \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ 0 & 0 & 0 \end{pmatrix} \oplus 0,$$

respectively, such that $b_{11} + b_{22} = 1$ and $b_{11}b_{22} = b_{12}b_{21}$. Then $\Phi(B_1)^s\Phi(A) + \Phi(A)\Phi(B_1)^s$ has operator matrix

$$\begin{pmatrix} 2b_{11} & b_{12} & b_{13} \\ b_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0 \quad \text{with spectrum equal to} \quad \sigma(B_1A^s + A^sB_1) = \{2a, 0\}.$$

It follows that (i) $b_{11} = a$, $b_{12}b_{21} = 0$, or (ii) $b_{11} = 2a$ and $b_{12}b_{21} = -4a^2$. If (ii) holds, then $b_{22} = b_{12}b_{21}/b_{11} = -2a$. But then $\Phi(B_1)$ has trace zero, which is impossible. Thus, (i) holds with $a = 1$. By Corollary 3.4, $\Phi(B_1) \neq \Phi(A)$, we may thus assume that with a space

decomposition $X_2 = \mathrm{span}\,\{x\} \oplus V_1$, $\Phi(B_1)$ has operator matrix

$$M_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{or} \quad M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Next, we turn to $\Phi(B_j)$ for $j = 2, 3$. Similarly, with a space decomposition $X_2 = \mathrm{span}\,\{x\} \oplus V_j$, we can assume that the operator matrix of $\Phi(A)$ has the form $[1] \oplus 0$, and $\Phi(B_j)$ have the form M_1 or M_2. Now, let $S_j = [1] \oplus T_j$, where T_j is an operator of changing a basis for V_1 to a basis for V_j. Then we see that the operator matrix of $\Phi(B_j)$ with respect to the space decomposition $\mathrm{span}\,\{x\} \oplus V_1$ has the form $S_j^{-1} M_1 S_j$ or $S_j^{-1} M_2 S_j$. Hence, with respect to the space decomposition $\mathrm{span}\,\{x\} \oplus V_1$, A has operator matrix $[1] \oplus 0$, and for $j = 1, 2, 3$, $\Phi(B_j)$ has operator matrix

$$\begin{pmatrix} 1 & X_j \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 \\ Y_j & 0 \end{pmatrix}, \tag{0.1}$$

where X_j, Y_j are rank one operators. But then there are two distinct elements $j, k \in \{1, 2, 3\}$ such that both $\Phi(B_j)$ and $\Phi(B_k)$ have operator matrices in upper or lower triangular form. It follows that

$$\sigma(\Phi(B_j)^s \Phi(B_k) + \Phi(B_k)\Phi(B_j)^s) = \{2, 0\} \neq \sigma(B_j^s B_k + B_k B_j^s),$$

which is a contradiction.

Next, suppose $a = b$. Then A has an operator matrix $\begin{pmatrix} aI_1 & T \\ 0 & aI_2 \end{pmatrix}$ for some operator T.

By claim 4, we see that $A^2 \neq 0$ and thus $a \neq 0$. Let B_1, B_2, B_3 have operator matrices $\mathrm{diag}\,(1, 0, 0) \oplus 0$, $\mathrm{diag}\,(0, 1, 0) \oplus 0$, $\mathrm{diag}\,(0, 0, 1) \oplus 0$. Then $\sigma(\Phi(A)\Phi(B_j)^s + \Phi(B_j)^s\Phi(A)) = \sigma(AB_j^s + B_j^s A) = \{2a, 0\}$. Using argument as before, we see that $\Phi(B_j)$ has operator matrix of the form (0.1), and there are distinct elements $j, k \in \{1, 2, 3\}$ such that $\sigma(\Phi(B_j)^s\Phi(B_k) + \Phi(B_k)\Phi(B_j)^s) = \{2, 0\} \neq \sigma(B_j^s B_k + B_k B_j^s)$, which is a contradiction. ∎

To prove Claim 8, Lemma 3.8 in the paper [1] should be replaced by the following.

Lemma 3.8. *Suppose* $\dim X \geq 3$. *Let* $P, Q \in \mathcal{I}_1(X)$. *Then* $PQ = 0 = QP$ *if and only if there is* $B \in \mathcal{B}(X)$, *which can be chosen to have rank 2, such that* $\sigma(PB + BP) = \{2, 0\}$, $\sigma(QB + QB) = \{-2, 0\}$, *and* $\sigma(BR + RB) = \{0\}$ *whenever* $R \in \mathcal{I}_1(X)$ *satisfies* $\sigma(PR + RP) = \sigma(QR + RQ) = \{0\}$.

Proof. Suppose P, Q in $\mathcal{I}_1(X)$ satisfy $PQ = 0 = QP$. Then there is a space decomposition for X such that P and Q have operator matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0 \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

Using the same space decomposition, let B have operator matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0$. Then B has rank 2 such that $\sigma(PB + BP) = \{2, 0\}$, $\sigma(QB + BQ) = \{-2, 0\}$. Consider any R in $\mathcal{I}_1(X)$

such that $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\}$. Using the same space decomposition as P and Q, we assume that R has operator matrix

$$\begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$$

where R_{11} is a 2×2 matrix . Since $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\}$, the $(1,1)$ and $(2,2)$ entry of R_{11} are both zero. Thus, R_{22} has trace one and rank one. We may then assume that R_{22} has operator matrix $[1]\oplus 0$. As a result, we may assume that the operator matrix of R has the form $\hat{R}\oplus 0$, where $\hat{R}$ or $\hat{R}^t$ has one of the following forms:

$$\begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & c & 1 \end{pmatrix} \text{ with } a = bc.$$

Consequently, $\sigma(BR+RB) = \{0\}$.

Conversely, suppose $P, Q \in \mathcal{I}_1(X)$ such that $PQ \neq 0$ or $QP \neq 0$. Then there is a space decomposition for X such that P has operator matrix $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \oplus 0$ and Q has operator matrix

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \oplus 0 \qquad \text{or} \qquad \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \oplus 0.$$

We assume that the former case holds. The proof for the other case is similar. Suppose there is a B in $\mathcal{B}(X)$ such that $\sigma(BR+RB) = \{0\}$ whenever R in $\mathcal{I}_1(X)$ satisfies $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\}$. Using the same space decomposition as P and Q, we assume that B has operator matrix

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

where B_{11} is a 2×2 matrix .

First, we claim that $B_{22} = 0$. If not, we may assume that the $(1,1)$ entry of B_{22} is nonzero. If R has operator matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \oplus 0,$$

we see that $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\} \neq \sigma(BR+RB)$.

Next, we claim that $B_{12} = 0$. If it is not true, we can find a suitable space decomposition for X such that the rank one block B_{12} has the form $\begin{pmatrix} 1 & 0 \\ 0 & T \end{pmatrix}$, where the last column is vacuous if $\dim X = 3$, and T has rank zero or one. But then if $R \in \mathcal{I}_1(X)$ has operator matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \oplus 0,$$

we have $\operatorname{tr}(BR+RB) = 2$ so that $\sigma(BR+RB) \neq \{0\}$. Similarly, we can show that $B_{21} = 0$.

Now, consider $B_{11} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Let R have operator matrix $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \oplus 0$. We see that $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\}$. Because $BR+RB$ has operator matrix

$$\left(\begin{pmatrix} b_{12} & 0 & b_{12} \\ b_{22} & 0 & b_{22} \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ b_{11} & b_{12} & 0 \\ b_{11} & b_{12} & 0 \end{pmatrix} \right) \oplus 0$$

with trace $2b_{12}$ and $\sigma(BR+RB) = \{0\}$, we see that $b_{12} = 0$. Since, $\sigma(PB+BP) = \{2, 0\}$ and $\sigma(QB+BQ) = \{-2, 0\}$, it follows that $(b_{11}, b_{22}) = (1, -1)$. Finally, for R with operator matrix $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \oplus 0$, we have $\sigma(PR+RP) = \sigma(QR+RQ) = \{0\}$. But then $BR+RB$ has operator matrix

$$\left(\begin{pmatrix} 0 & -1 \\ 0 & -1-b_{21} \end{pmatrix} + \begin{pmatrix} -b_{21} & 1 \\ b_{21} & -1 \end{pmatrix} \right) \oplus 0 = \begin{pmatrix} -b_{21} & 0 \\ b_{21} & -2-b_{21} \end{pmatrix} \oplus 0,$$

which cannot be a nilpotent. ■

References

[1] J.C. Hou, C.K. Li and N.C. Wong, "Maps preserving the spectrum of generalized Jordan product of operators", Linear Algebra Appl. 432 (2010), 1049-1069.

(Hou) Department of Mathematics, Taiyuan University of Technology, Taiyuan, 030024, P. R. of China

E-mail address: houjinchuan@tyut.edu.cn

(Li) Department of Mathematics, The College of William & Mary, Williamsburg, VA 13185, USA. Li is an honorary professor of the University of Hong Kong

E-mail address: ckli@math.wm.edu

(Wong) Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung, 80424, Taiwan.

E-mail address: wong@math.nsysu.edu.tw